THE FUNDAMENTALS OF ELECTRO-MAGNETISM

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BY

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To the memory of

ERNEST *and* EDITH CULLWICK

this book is fondly dedicated

by their son

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

NEWTON

PROLOGUE

The known facts of electricity and magnetism form an exact and coherent body of knowledge of surprising beauty and symmetry, and the unravelling of this beauty by patient thought and experiment forms one of the most fascinating stories of all time. From small and disjointed beginnings, from lodestone and amber, this most incorporeal of nature's secrets gradually capitulated to the restless mind of man, until at last the knowledge handed on by Oersted and Faraday, by Ampère and Maxwell, bids fair to embrace the whole of the physical universe.

To the electrical engineer falls the work of applying this knowledge to practical use. No matter what the particular application, be it a dynamo, a telephone, a broadcast concert, a cathode-ray oscillograph, or a toy magnet, the fundamental physical phenomena are common to all, and fit into one coherent and unified system.

Now it appears almost as though the usual education of the electrical engineering student has been specially designed to make it difficult for him to grasp the essential unity and coherence of his subject. Two hundred years ago electricity and magnetism were regarded as distinct phenomena, but that is scarcely a sound reason for introducing them as such to the youth of the twentieth century. The three fundamental and primary concepts of length, mass, and time present no difficulty to the average student, since he has been aware of them, though perhaps unconsciously, in his everyday life almost since he was born, and this unconscious preparation makes the understanding of the principles of mechanics relatively easy. The understanding of electrical phenomena, however, must be based upon familiarity with a fourth primary physical concept, with which the student usually has no first-hand experience when he begins his studies. It appears as though he is first born into the mechanical world and later, when he starts the organized study of science, into the world of electricity, and

only when he has served a certain apprenticeship in the methods and philosophy of science can he accept the reality of an electric charge on the same physical level as length, mass, and time.

How confusing, therefore, must it be for him if he is not told at once what this fourth physical concept is. If he is introduced to electric charges, magnetic poles, electric and magnetic fields as though they were all equal in the possession of physical reality, little wonder that they become in his mind all equal in mystery. And if he is then told, in dimensional formulae, that none of these is to be linked with mass, length, and time in the quaternion of primary concepts, but that he must use *either* the "permittivity" *or* the "permeability" of free space, the race is indeed for the swift, and the battle for the strong!

During some little experience as a teacher, I have become more and more aware of certain reasons why the average student's knowledge is so often disjointed and confused. These are, briefly, the stretching of the mathematical analogy between electricity and magnetism into the physical realm (which leads to magnetic definitions in terms of impossible experiments on non-existent unit poles); the duality of the electro-static and electro-magnetic systems of units (which often leads to the ignoring of electro-statics and electric fields by engineers who should know better); and the confusion of thought existing as to the physical reality of electric and magnetic fields and "lines of force" (which leads to endless discussions about the "mechanism" of electro-magnetic induction, and the physical dimensions of μ_0 and κ_0).

At first I tried to maintain a humble attitude of mind, and a belief that one day I would understand the necessity for these mental burdens, but certain recent influences have destroyed my humility and have made of me an irreverent sceptic. At an Oxford Summer School for Engineering Teachers I heard it suggested, by an apparently well-qualified teacher, that electro-statics might very well be omitted from electrical engineering courses—presumably because, in dealing with it, one had to shut off that portion of one's brain which contains the electro-magnetic system, and to open up the portion which

contains the poor half-forgotten electro-static units. Further, a senior Inspector of the Board of Education uttered the astounding heresy of renouncing the unit magnetic pole.

The mysteries of lines of force and of electro-magnetic induction form a fruitful source of unlimited philosophical speculation, to which pleasant pursuit I was introduced by an interesting but inconclusive correspondence, between my seniors, in the technical press. The result was a temporary sojourn with the poets.

> Myself when young did eagerly frequent
> Doctor and Saint, and heard great argument
> About it and about: yet evermore
> Came out by the same door wherein I went.

My gradual rescue from this plight was completed upon reading P. W. Bridgman's *The Logic of Modern Physics*,* the thesis of which is that the modern philosophy of physical knowledge, originating with Einstein, takes the meaning of a concept as synonymous with the operations (actual experiments or theoretical calculations) by which its measure is determined, and that the proper definition of a concept must be in terms of such real operations. Divorced from, or outside the range of, the appropriate operations, the concept can have no meaning. It may be suggested, by metaphysicians, that this "operational viewpoint" is no more than a refuge in which modern physicists preserve their sanity, but to me this book was a tremendous help in disentangling fact from fancy.

It appears to me that a problem of supreme importance now faces all teachers of electrical engineering (and of the preparatory physics courses), a problem whose solution necessitates a radical change of viewpoint, and methods of presenting the fundamentals of electricity and magnetism. The conventional viewpoint of the teaching of to-day is still based upon the ideas of Faraday and Maxwell, to whom the existence of a material energy-transmitting medium, or aether, was a fundamental necessity. Since the days of the Michelson-Morley experiment, however, physicists have formulated, and

accepted, the theories of relativity and of the quantization of energy, with which the older theory is not consistent. Teachers of electrical engineering are sometimes forced to dip into these new theories, as for instance when discussing the motion of electrons in a high-voltage cathode-ray oscillograph, or the fundamental laws of photo-electric cells, but apparently no serious attempt has yet been made to present the elementary theory in a way which is consistent with the viewpoint of modern physics.

This book has been written with the aim of making some contribution, however imperfect, to the solution of this problem. In such an attempt, the physical viewpoint which appears to be most effective, and free from unnecessary artificialities, is that of "action at a distance, retarded in time". That is, we look upon the mutual actions of electric charges as the basis of the phenomena, and forsake the attempt to "explain" the mechanism of these effects by means of the properties of an electro-magnetic field whose physical existence cannot be proved. Electric and magnetic fields are retained, not as physical realities, but as extremely useful mathematical vector concepts, and to these we add, in exactly the same category, the vector potential.

Now this scepticism as to the physical reality of electric and magnetic fields may appear so startling, and so contrary to the reader's accepted views, that he will now throw the book down with a snort of impatience at my disrespect for authority. If this is so, then I must bring Bridgman to my aid, and quote what he says on the subject:

Now nearly every physicist takes the next step, and ascribes physical reality to the electric field, in that he thinks that at every point of the field there is some real physical phenomenon taking place which is connected in a way not yet precisely determined with the number and direction which tag the point. At first this view most naturally involved as a corollary the existence of a medium, but lately it has become the fashion to say that the medium does not exist, and that only the field is real. The reality of the field is self-consciously inculcated in our elementary teaching, often with considerable difficulty for the student. This view is usually credited to Faraday, and is considered the most

fundamental concept of all modern electrical theory. Yet in spite of this, I believe that a critical examination will show that the ascription of physical reality to the electric field is entirely without justification. I cannot find a single physical phenomenon or a single physical operation by which evidence of the existence of the field may be obtained independent of the operations which entered the definition. The only physical evidence we ever have of the existence of a field is by going there with an electric charge and observing the action on the charge (when the charges are inside atoms we may have optical phenomena), which is precisely the operation of the definition. It is then either meaningless to say that a field has physical reality, or we are guilty of adopting a definition of reality which is the crassest tautology.*

If, however, such a viewpoint is adopted, many problems begin to lose their complexity, for we now cease to worry about "mechanism" and concentrate upon the physical facts. If, following Einstein, we take the invariant finite speed of propagation of electrical effects, not as the result of an elaborate theory of wave-propagation in the aether, but as an experimental fact which should be accepted as a fundamental postulate, the whole subject then comes within the range of engineering undergraduates whose mathematical equipment does not include vector analysis. In addition, I firmly believe that such a method encourages a co-ordinated physical insight which forms a sound basis for more advanced study by means of vector analysis, which too often becomes merely a mathematical exercise with no accompanying appreciation of the underlying physical concepts. Further, by accepting the meaninglessness of absolute velocity, we are able to treat electro-magnetic induction in a rational manner which is free from ambiguity and artificiality.

Fortunately, there is no need to incorporate wave-mechanics or quantum theory into our analysis. We are dealing with large-scale or macroscopic phenomena, and it is only in the realm of the very small, such as inside the atom, that the certainties and continuities of classical theory break down into uncertainty and discontinuity. Where necessary, as in the theory of the photo-cell, we may accept the facts of the

* *The Logic of Modern Physics*, p. 57.

quantization of energy, but since we have already admitted our ignorance of the mechanism of all electrical effects, one theory need not puzzle us any more than the other. A truly comprehensive physical theory must of course embrace both microscopic and macroscopic phenomena, and must include the transition zone between quantum and classical theory, but until such a theory is available in a simple form we must be content with the present situation.

Using this method, we have to find new definitions for many quantities, such as H and B, and no use is made of the unit magnetic pole. The foundation of the electro-magnetic system upon this magneto-static concept leads to a strange inversion of reality, wherein electric charge appears as one of the most hypothetical of all the concepts, instead of the most fundamental. An acceptance of electric charge, and its invariance at rest or in motion, is surely more consistent with present physical knowledge and theory, and leads to no artificiality in the subsidiary definitions. For instance, the present definition of the "international" ampere in terms of electrolytic deposition is completely consistent with this method.

The book has no pretensions to completeness. Its primary function is a simplification and unification of fundamental ideas, and a logical presentation of electro-magnetic theory in a manner consistent with the physical viewpoint. I merely hope that it may be helpful to the thoughtful student or teacher, who may be interested in a method different from that with which he is familiar.

It is not my opinion that a sudden and complete change, either in units or in methods of approach, is possible in undergraduate teaching, but in discussing these matters with my friends during the past few years I have gained the impression that there is sufficient dissatisfaction with the older methods, and confusion of thought due to their use, to justify the present attempt at clarification.

ACKNOWLEDGMENTS

To all who have helped me in this work, whether by their books, papers, or stimulating conversation, I wish to express my deep indebtedness. In addition to references in the text, I particularly wish to mention:

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NOTE ON THE SECOND EDITION

Since the author has been absent from academic work for a long period since this book was first published in 1939, changes in the present edition are few. Some inaccuracies have been corrected, some points clarified, and various appendages of doubtful value have been removed.

Some apology is needed for the sudden appearance, in Chapter V, of the methods of vector analysis in dealing with the electric field of a moving magnet. This section can, of course, be omitted by undergraduate students but is included in order to obtain an important result which is often taken for granted.

The m.k.s. system of units has found increasing favour during the past decade, both by electrical engineers and by physicists. It has been adopted with enthusiasm, usually in the "rationalized" form, by many of those who took part in the great war-time development of micro-wave techniques, and its use in teaching is now commonplace.

The author's thanks are extended to the reviewers of the first edition for their constructive and helpful criticism.

E. G. C.

Ottawa, Canada

NOTE ON THE THIRD EDITION

In July 1950 the Advisory Committee on Electric and Magnetic Magnitudes and Units (Technical Committee No. 24) of the International Electrotechnical Commission decided in favour of the rationalized form of the m.k.s. system of units, and in 1956 this decision was finally ratified by the I.E.C.

Consequently, the second edition of this book, in which the unrationalized form of the Giorgi system was used, has for a long time been out of step with current usage. I am therefore most grateful to the Syndics of the Cambridge University Press for their decision to publish this new edition, in which the rationalized system of units is used throughout and some sections of the text have been revised.

E.G.C.

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CHAPTER I

THE ELECTRO-STATIC FIELD AND THE ELECTRIC CURRENT

INTRODUCTION

ELECTRIC CURRENTS AND ATOMS

In electrical engineering use is made of the energy transformations associated with the flow of electric currents under various conditions, so that an exposition of the fundamentals of the subject must resolve itself into an account of electric currents and the circumstances under which transformations of energy can take place.

An electric current is a movement of electric charges, and every electric charge is usually considered to be attended by a condition in the surrounding space which is called an "electric field". Thus we may expect the properties of electric charges in motion to extend into the space around the conductor which carries the current. These properties include what is known as "magnetism", whereas the study of charges at rest is commonly called "electro-statics". The complete theory which covers the general case of moving charges is that of "electro-magnetism" or "electro-dynamics". In addition, however, a very complete theory of "magnetism" has been developed in which no account of its real meaning is taken in postulating the fundamental concepts; useful and firmly established as this theory is, it must be borne in mind that magnetism divorced from electricity has no physical reality, and the "field" of a permanent magnet is only an aspect of moving electrons in the iron.

It is now generally accepted that material bodies consist of atoms which in turn are arrangements of entities, some or all of which have equal and indivisible electric charges. This fundamental and indivisible charge will be called the "elec-

tronic charge" in this book, and according to recent measurements is equal to:

$$4{\cdot}803 \times 10^{-10} \text{ electro-static unit}$$

or $$1{\cdot}602 \times 10^{-20} \text{ electro-magnetic unit}$$

or $$1{\cdot}602 \times 10^{-19} \text{ coulomb.} \qquad (1)$$

The entities having a negative electronic charge have been found to have a mass of $9{\cdot}11 \times 10^{-31}$ kg., and it may be that the whole of this mass is due to the charge itself. In this work the name "electron" will be used for these negative charges, and it is electrons that provide our chief source of electric current. The mass due to the charge on the surface of a sphere can be calculated,* and is inversely proportional to the radius of the sphere. The "equivalent charged sphere" having the same charge as an electron, and having a "charge mass" equal to the known "rest mass" (the mass it appears to possess when its velocity is less than about one-tenth that of light), has a radius of $1{\cdot}9 \times 10^{-13}$ cm. This is our nearest approach to a knowledge of the "size" of an electron, but we do know from experiment that, when it behaves as a particle, it must be inconceivably small.

The common conception of the structure of an atom is that of a number of electrons (the orbital or shell electrons) moving in some way about a central nucleus having a positive charge. Up to a few years ago, it was thought that atomic nuclei themselves consisted solely of electric charges, a number of protons (positive electronic charges having a mass about 1840 times that of an electron, and so proportionally smaller) being associated with a smaller number of electrons. At that time, therefore, it appeared as though all matter consisted of electric charges, but more recent discoveries have shown that this is not necessarily true. There is to-day the possibility of an inconveniently large number of elementary particles, some having unit electronic charge and some having no charge at all. It seems unlikely that this state of affairs is final, and the reality or indivisibility of neutrons, positrons, neutrinos, or negatrons need not greatly concern us. If, however, the present

* See Chapter III, Section 29, and foot-note on p. 179.

view that the fundamental units of the nucleus are *neutrons* and *protons* is correct, where the neutron is an elementary *uncharged* particle of about the same mass as a hydrogen atom, and the proton is perhaps a neutron tied to a positron (a positive electron), then it may mean that matter is by no means purely electrical. It seems as though physics progresses by the encouraging sight of a possible finality on the horizon, which when reached merely opens up a further stretch of unexplored country beyond.

In metals the distance between adjacent atomic nuclei is of the order of 4×10^{-8} cm., which may be taken as a rough indication of the "diameter" of an atom (the Rutherford-Bohr model of atoms gives diameters ranging from about 1×10^{-8} to 6×10^{-8} cm.). If this is so, and if the electron, considered as a spherical charge, is magnified to the size of a golf ball, then an atom on the same scale would have a diameter of about two and a half miles. Furthermore, the nucleus itself, carrying as it does practically the whole of the atomic mass, does not appear to be greater, in the heaviest atoms, than about one ten-thousandth of the size of the atom. Apparently, then, the atom is a very loosely constituted aggregation of matter, and consists mostly of free space.

The material universe contains at least 92 elements, each having a different positive charge in its atomic nucleus. It is this positive nuclear charge which determines the chemical properties of an element, and the value of this charge, in electron units, gives the *atomic number*. If this number gives the number of protons in the nucleus, then the number of neutrons will be roughly equal to the difference between the atomic number and the atomic weight. That this relation is only approximate is due to the convertibility of mass and energy, so that some of the mass of the nuclear constituents is converted into "binding energy". *Isotopes* of an element have the same nuclear charge, and hence the same chemical properties, but different atomic weights. Apparently they differ in the number of nuclear neutrons. A neutral or normal atom has no aggregate electric charge, so that the number of orbital electrons must be equal to the charge, in electron units,

of the nucleus. Thus according to this theory the numbers of electrons and protons in a neutral atom are equal.

Atoms may be deprived of one or more of their outer electrons, and are then said to be "ionized", or to be "ions" They will now possess a positive charge equal to the charge of the electrons removed. Other atoms have the property of readily adopting additional electrons, becoming negative ions. Radioactive elements, however, owe their properties to the instability of their nuclei, from which they occasionally expel particles, and in so doing degenerate eventually into elements lower down the atomic scale.

Electrons and protons attract one another, and the stability of the atom may be considered to be partly due to the balance of these attractions and dynamic forces. Electrons are repelled by electrons, protons by protons, these forces between electric charges being summarized in the rule, formulated long before the discovery of the electron, that "like charges repel", and "unlike charges attract".

To produce an electric current, it is first necessary to have electric charges free to move, and in conduction currents these are usually electrons, positive ions, or both. Substances may in general be divided into two classes: conductors and insulators. In conductors (e.g. metals) we imagine a very large number of electrons which have considerable freedom of motion through the atomic structure, while in insulators such free motion is extremely restricted: in the perfect insulator it would be zero.

In electro-static theory, the force between charged bodies at rest is "explained" by the hypothesis that the space surrounding a charge is in an abnormal state which is called an *electric field*, and that another charge placed in this field experiences a force. The magnitude and direction of the electric field at any point are defined by the magnitude and direction of the force which a small charged body experiences when situated at that point.

The magnitude of a charge is defined in terms of the mutual force experienced by two similar charges when situated a definite distance apart, and if the charges are imagined to be concentrated at dimensionless points the law of mutual force

is the familiar inverse-square relation. The success of this hypothesis in giving a theory consistent with experiment made the concept of a point charge one of considerable value, though as we shall see later it is inconsistent with classical theory. Although the *magnitude* of a charge is a primary physical concept, its *configuration* is merely hypothetical, a fact which is used in introducing the concept of electrical displacement.

It was really to avoid the philosophical difficulty of accepting "action at a distance" that the further concept of the electric field was first introduced. It was felt that if a charged body experiences a force due to the proximity of another charged body, then that force must be transmitted through the intervening space by some property of that space. Since an electric field can be detected only by the behaviour of charged bodies, there is no experimental proof of its existence in the absence of electric charges. It is true that the concept of energy storage in the field, a concept which fits in well with the phenomena of electro-magnetic waves, gave colour to the idea that an electric field must have physical reality (that is, it was a *strain* in the *aether*), but modern ideas of the quantization of energy seem to have invalidated this argument. However, in reducing the facts of electro-magnetism to a consistent theory, we make use of many hypothetical concepts, which are of great value; but we must not fall into the common error of endowing these hypothetical concepts with physical reality.

PART I

THE ELECTRO-STATIC FIELD

1. Coulomb's inverse-square law.

The physical phenomena associated with stationary bodies which carry constant electric charges are all consistent with *Coulomb's Law*, which states that the mutual force between two stationary and isolated point-charges acts along the line joining them, is proportional to the product of the two charges q_1 and q_2, and is inversely proportional to the square of the

distance between them. The law can therefore be expressed in the form

$$F=\frac{q_1 q_2}{kr^2}, \tag{2}$$

where k is a constant depending upon the system of units.

In the c.g.s. system, F is in dynes and r in centimetres and if the constant k is given the value of unity then the charges are measured in c.g.s. electrostatic units. Now the field of an isolated point-charge has spherical symmetry, in which case we might reasonably expect to find the expression for the field to contain the spherical factor 4π. In the c.g.s. electrostatic system, however, this is not so, and the factor 4π turns up where one would not reasonably expect to find it, such as in the expression for the capacitance of a parallel-plate capacitor, and in all cases of spherical symmetry the factor is absent. For this reason such a unit-system is said to be *unrationalized.*

In the m.k.s. system of units we may, if we wish, use an unrationalized form of electro-magnetic theory, but the system now universally adopted is that in which the factor 4π appears more appropriately and which is called the *rationalized m.k.s. system.* We then write Coulomb's Law in the form

$$F=\frac{q_1 q_2}{4\pi\epsilon_0 r^2}, \tag{2a}$$

where F is in newtons, q_1 and q_2 in coulombs, and r in metres. The constant ϵ_0 is usually called the permittivity of free space, but we shall use the name *primary electric constant* as being more consistent with the viewpoint of this book. Its value is $8{\cdot}854\times10^{-12}$ coulomb/volt-metre, or farad/metre, and a convenient practical form of the law in m.k.s. units is

$$F \doteq 9\times10^{9} q_1 q_2/r^2 \quad \text{newtons.} \tag{2b}$$

Coulomb's Law is independent of any "field concept", and consequently might be called an "action at a distance" formula. If we now introduce the concept of the electric field as the force-transmitting link, we define the intensity $\boldsymbol{E}$ of the field, at the point where the charge q_2 is situated, by putting

$$\text{force on } q_2 = F = Eq_2. \tag{3}$$

Equation 1(3) therefore gives the force experienced by a

charge q_2 when situated in an electric field of intensity E. This equation incorporates the sole definition and meaning of an electric field: that is, whenever an electrically charged body experiences a force in virtue of its charge, as distinct from any mechanical forces independent of its charge, we may say that the body is situated in an electric field, whose value is *defined* by equation 1(3).

In the particular case which we are considering, in which the source of the field is a stationary charge, we call the field an *electro-static* one. We can give an electric field the particular names *electro-static* or *electro-magnetic* only if we have complete knowledge of its origin: the test charge q by which we measure its intensity can make no such distinction, and the total force which it experiences may be due to a combination of causes, the electric field of each cause being added vectorially to that of every other in order to produce the total field. This *Principle of Superposition* is of great importance, and may be stated thus:

> If the total force on a charge is due to a variety of causes, then it may be calculated by summing, vectorially, the forces calculated by considering each cause independently.

To find the intensity of the electro-static field due to a point charge q_1, at a point distant r from it, we equate 1(2) and 1(3): and get

$$E = \frac{q_1}{4\pi\epsilon_0 r^2}. \tag{4}$$

We say then, that E is the intensity of the electro-static field at a point in space distant r from an isolated point-charge q_1, but we endow E with no meaning other than that of a mathematical vector which enables us to simplify the calculation of the force experienced by the "test-charge" q_2. In order to measure E, test-charges of some sort must be present, and the experimental facts are no more than the forces, and the consequences of these forces, which the test-charges experience. Of the "mechanism" which causes these forces we are, at present, entirely ignorant, and although the hypothesis of the electric field is of enormous value to us in developing a

working theory, it should never be forgotten that it is only a hypothesis, and that from the operational viewpoint it does not pass the test of physical reality.

Electric charge, on the other hand, is defined in terms of a real experiment, in which the presence of nothing but material bodies is presupposed. Of the ultimate essence of electric charge it is true that we know nothing, and our knowledge of it is confined to a knowledge of experimental phenomena, with which our theory of the interaction of electric charges is consistent. If we are asked: "What is an electric charge?" we can do no more than state the experimental definition of unit charge:

> Unit charge, or quantity, of electricity, is that charge which, when considered to be concentrated at a dimensionless point, and at rest, and unit distance from a similar stationary charge, both charges being remote from any other matter, experiences a force of $1/4\pi\epsilon_0$ units.

There is good reason to suppose that electric charge is an ultimate physical reality, of which all material bodies are composed. Consequently it is a fundamental starting-point in our knowledge of the physical world, and to attempt to explain it in terms of more familiar (but more complicated) concepts is merely to think in circles.

2. Lines of force and lines of flux or displacement.

The intensity, E, of an electric field is a vector quantity, since it has direction as well as magnitude. The simplest method of depicting its configuration is a graphical one: drawing a number of lines which show the paths which point-charges would follow if free to move, singly, under the influence of the field. As the field is three-dimensional, however, a two-dimensional diagram of this kind can show only the configuration of the field in any one plane. Thus in the case of an isolated point-charge the direction of the field at any point is shown by the direction of straight lines radiating outwards in all directions from the charges (Fig. 1), so that we must have a three-dimensional picture or model.

If the number of such lines issuing from an isolated charge q is arbitrarily made equal to q/ϵ_0, then the number of these lines passing through unit area of a concentric sphere of radius r is equal to the field intensity E, for

$$\text{Lines (of force) per unit area} = \frac{q}{\epsilon_0 4\pi r^2} = E,$$

and in this case the lines are called "lines of force" or "lines of E".

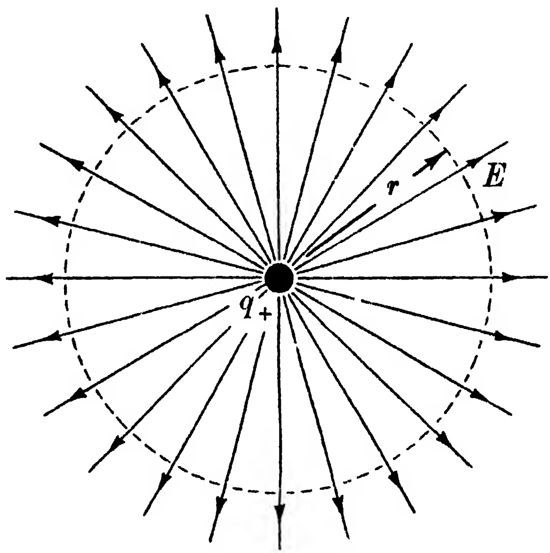


Fig. 1. Isolated charge

Now if the charge q is surrounded, not by free space or air. but by a solid homogeneous insulator, the electric field intensity E inside this solid medium, and due to q, cannot be measured directly, since we cannot perform the experiment of measuring the force on a test charge embedded in the material. Actually we arrive at a computation of E (i.e. the force experienced by a unit charge so embedded) by indirect methods, which are validated by the agreements between theory and experiment which result, and we find in such cases that E is now dependent upon the type of insulation used. This matter is discussed more fully in Section 4 following, under "dielectric constant".

It is convenient, as will be seen later, to introduce an additional vector quantity, which in homogeneous isotropic dielectrics is always proportional to E and codirectional, but having the property that the number of its "lines" issuing from a given static charge q is *dependent upon q and upon q alone.* If we arbitrarily fix the number of such lines issuing from an isolated static charge q to be equal to q we call them *lines of electric flux or displacement.*

The number of such lines of electric flux, per unit area of a concentric sphere of radius r, is now, in free space,

$$D = \frac{q}{4\pi r^2} = \epsilon_0 E \tag{5}$$

D is called the *displacement density* or the *electric flux-density* of the field, and the total electric flux, ψ, is clearly $4\pi r^2 D$ and is equal to the charge q from which it emanates. Like q, it is measured in *coulombs*. Flux-density D is therefore measured in *coulombs per square metre.*

The concept of electric flux is introduced here because it is fundamental to Maxwell's theory of electro-magnetism. It is further discussed in Section 6 of this chapter.

Tubes of electric flux. A disadvantage of the conception of lines of electric flux lies in the graphical discontinuity *between* the lines. A method of dealing with electro-static fields which does not suffer in this way is to use *tubes* instead of lines: we imagine the whole of the space in which the electric field exists to be made up of contiguous tubes, the axes of which are everywhere coincident with lines of electric flux. If, again, we make the number of such tubes issuing from a charge q equal to q, the number of tubes per unit area of a surface normal to the field will be equal to D. Thus if A_n is the area of a normal section of such a unit tube at any point, we have

$$DA_n = 1 \text{ coulomb.} \tag{6}$$

A tube of unit electric flux thus has its origin on a charge of unit value, and terminates (as will be proved later) on a charge of equal value but of opposite sign.

3. Work and potential: potential difference (p.d.).

Since a charge experiences a force when situated in an electro-static field, it will perform work if it is allowed to move under the influence of this force. In other words, it possesses potential energy. If the charge has unit strength, the work involved in its motion between two points in the field, A and B, is termed the *potential difference* (p.d.) between the two points. Further, if work is done *by* a positive charge in moving from A to B, we say that the point A is at a higher electro-static potential than B. Thus we say that a positive static charge is at a higher potential than a negative charge, and further we take the positive direction of the electric field (or of a line of force) to be that direction in which a free positive charge will tend to move. There is therefore a fall of electro-static potential in the positive direction of an electro-static field.

Absolute electrostatic potential. The absolute electrostatic potential at a point in an electrostatic field may be defined as the work done by a unit positive charge in moving, under the influence of the field, from the given point to an infinite distance. The potential at a point distant R from an isolated point-charge q is therefore

$$U = \int_R^\infty E\,dr = \int_R^\infty \frac{q\,dr}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 R}. \tag{7}$$

In 1(7) it is assumed that the unit charge moves along a line of force. If it moves along any arbitrary path in the field, the work done in moving a short distance δl will be $E\,\delta l\cos\theta$, where θ is the angle between **E** and **δl**, and this may be expressed as the *scalar product* $\mathbf{E}\cdot\mathbf{\delta l}$ of the two vectors **E** and **δl**. But since $\delta l\cos\theta = \delta r$ it follows that the result given by 1(7) is independent of the path chosen. From the Principle of Superposition it follows that this is true for the potential due to any static distribution of charge. It should be noted that U is a *scalar* function, and may therefore be called the *scalar potential.*

Potential difference as line-integral of field intensity. In practice we are concerned with the difference between the potentials at different points in the field, rather than with absolute values of potential. It follows from the above that

the fall of potential from any point A to another point B is given by

$$V_{AB} = U_A - U_B = \int_A^B \mathbf{E} \cdot \mathbf{dl}, \tag{8}$$

that is, the *line-integral* of the field intensity **E** along any path between A and B. The symbol V may also be used to denote the *rise* of potential between two points, and in either case it denotes a *potential difference*, or p.d.

Potential gradient as a measure of field intensity. If, in 1(8), the points A and B lie on a line of force which is in the direction A to B, and are separated by a very short distance δl. then the *increment* of potential from A to B is $\delta U = -E\,\delta l$, so that, in the limit,

$$E = -\frac{\partial U}{\partial l}, \text{ the \textit{potential gradient}.} \tag{9}$$

The minus sign reminds us that the potential *falls* as we progress in the positive direction of E

Definition of unit p.d. It follows that

> Unit p.d. exists between two points in an electro-static field when unit work is involved in the motion of unit charge between the two points.

[Substituting the practical units *joule* (work) and *coulomb* (charge) in this definition, we obtain the definition of the *volt*, the practical unit of p.d. From (9), then, we see that field intensities can be measured in *volts per unit length*. Sometimes the *volt per cm* is used, but the *absolute* practical unit (see Appendix on the m.k.s. system) is the *volt per metre*.]

"Potential gradient" strictly applies to electro-static fields only (i.e. fields having an electro-static potential), but the term "potential" is often used by the engineer as synonymous with "voltage", so that "potential gradient" is often used, erroneously, to denote the intensity of any electric field, regardless of its origin.

Equipotential surfaces. Any surface normal at every point to the lines of force of an electro-static field is an *equipotential surface*, since no work is involved in moving a charge between two points on the surface. Consider the potential difference between any two points, such as P and Q (Fig. 2), in an electro-static field. Whatever path we take in moving a unit charge

from P to Q, the same amount of work will be involved: i.e. that corresponding to the difference of potential between the equipotential surfaces A and B We may take a simple path. (1) in Fig. 2, in which we travel from P along a line of force until we meet the equipotential surface B, from which point we travel to Q by a path which lies in this surface. Only the motion along the line of force will involve work. On the other hand, we may take a more complicated path such as (2), in which case work will be involved in the motion of the charge at all points of the path where the electric field has a component in the direction of motion. The total work involved in the operation will be the same as before

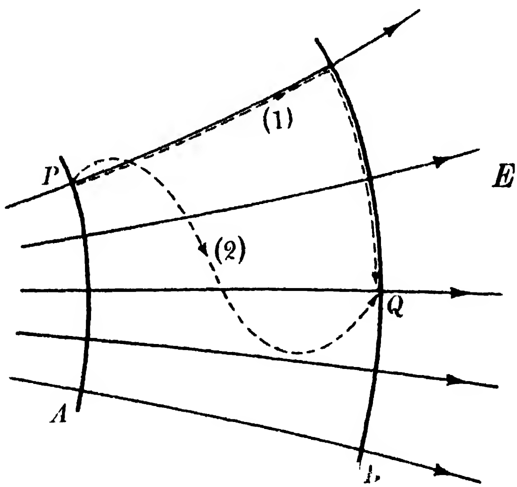


Fig 2 Equipotential surfaces

It follows immediately that the p d. around any closed path in an electro-static field must be zero, for, starting at any point in the path, we start from a definite level of potential to which we return when we complete the circuit i e

$$\oint \mathbf{E}_s \cdot \mathbf{dl} = 0. \tag{9a}$$

These properties of the electro-static field, expressed in terms of *potential*, are logical deductions from Coulomb's inverse-square law (i.e. the electro-static field obeys the inverse-square law). Electric fields produced by changing

currents or by moving magnets or current-circuits (i.e. by *electro-magnetic induction) do not necessarily obey the inverse-square law,* and in such fields the line-integral of E* taken between two points is no longer independent of the path taken, while the work involved in the motion of a charge in a complete circuit is not necessarily zero.

4. Charged bodies.

Coulomb's inverse-square law is based upon the conception of charges concentrated at dimensionless points. Such a conception is approximated in reality by the electron, but in practice we have to deal with bodies of normal dimensions which contain inconceivably large numbers of these fundamental charges. Let us consider some of the well-known facts of electro-statics, and see how they may be reconciled with a simple picture of the electronic structure of matter

A. *Charge on an isolated conductor.*

Consider first a conductor which has equal numbers of positive and negative electronic charges (i.e. it is what is commonly called an *uncharged body*). Some of the negative charges (electrons) may be considered to be moving freely within the structure of the conductor, in random directions. For the total number of positive and negative charges to balance, there must be a number of atoms deprived of electrons, these atoms being fixed (except for thermal agitations) in the crystal structure of the solid conductor, becoming positive ions. If the conductor is not situated in an external electric field, the mobile electrons and positive ions must be distributed uniformly throughout the material, for the mutual forces of all these charges must balance No portion of the conductor, therefore, will possess any aggregate charge.

Now suppose that the conductor is given a static charge. That is, a number of the free electrons are removed (it then becomes positively charged) or a number of free electrons are added (a negative charge). Suppose first that the "charge" distributes itself uniformly throughout the mass of the conductor so that each unit volume has the same aggregate charge. Then the charges on all these elements of volume will repel

each other, and since a conductor is a body in which charges can move more or less freely it follows that, if these charges can move, they will move outwards as far as they can go, i.e. to the surface. If the aggregate charge is negative (a surplus of electrons) we then think of the surplus electrons repelling each other until they are all situated at the surface of the body, while if the charge is positive (a dearth of electrons) we think of the positive ions in the interior attracting the mobile electrons from the surface, leaving the latter with a positive charge, very "thinly spread". We have the well-known result that a static charge on a conducting body is *limited to the surface.* If the conductor is a hollow shell, then all the charge is on the *outer* surface. It also follows that, if empty, there can be no electro-static field inside a hollow shell, since, if there were, its lines of force would end on charges on the inner surface.

There can therefore be no electric field *inside* a conductor which carries a static charge.* By the same reasoning we see that the surface of a conductor must be an *equipotential surface*, for any difference in potential between two points on the surface would cause a motion of the mobile electrons until the potential difference had disappeared.

The charge, however, sets up an electric field *exterior* to the conductor which, since the surface is equipotential, must leave the surface normally.

B. *Induced charges on conductors.*

Now suppose a conducting body (B in Fig. 3), which may or may not possess an aggregate charge, to be situated in an external electric field due to other charges (represented by A). The free electrons in the conductor B will move under the influence of the field of A, and in the case shown the end of B near to A will become negatively charged, while the remote end, being deprived of electrons, will attain a positive charge.

* If we imagine that *all* the mobile electrons, which distinguish a conducting body, could be removed, then of course an electric field could exist inside the body, but it would no longer be a conductor. In any practical case, the actual number of electrons removed or added, in giving a charge to a body, is extremely small compared with the number of mobile electrons in the conductor.

The field intensity along the surface of B will again be zero, so that the lines of force will enter normally to the surface. We can think of this result as being due to the cancellation, along the surface of B, of the field due to A by the field of the "induced" charges on B.

Now the tubes of force from any charged body end somewhere in an equal and opposite charge, so that the total negative charge induced on neighbouring bodies by a positively charged body is equal to the total positive charge on that body. In the case of Fig. 3 the diagram is incomplete: the

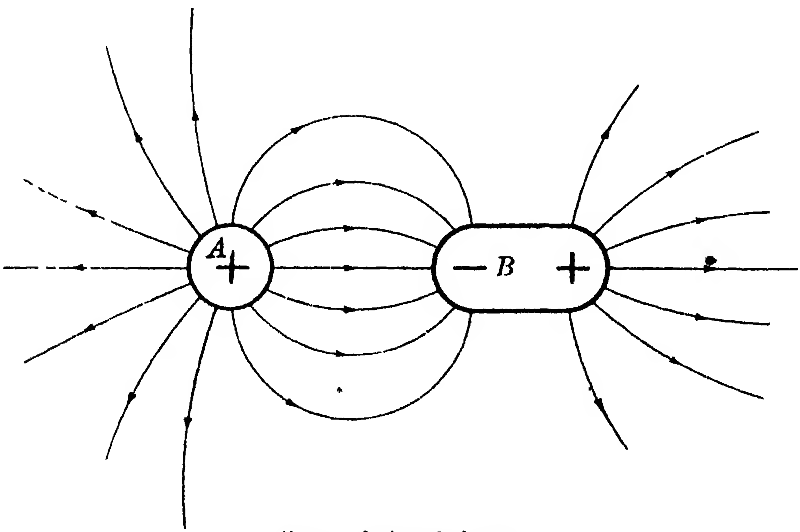


Fig. 3. Induced charges

electrons which were removed from A, in the process of charging it, must be somewhere, so that we cannot imagine the bodies A and B to be completely isolated in space. Some of the tubes of force from A and B will terminate on negative charges on the surfaces of the bench or the walls of the room.

In Fig. 4 a positively charged body is shown suspended, by an insulating thread, inside a hollow conductor, and in this case all the lines of force from the charge terminate on the inside surfaces of the enclosing box. The total induced negative charge on these inner surfaces is now exactly equal to the positive charge on the suspended body. If the box has no

aggregate charge, then an equal positive charge will be induced on its outer surface, the lines of force from which will terminate somewhere on an equal negative charge. In the whole universe the total amounts of negative and positive electric charges are apparently equal.

C. *Charges on insulators · dielectric constant.*

In the ideal insulator, free charges cannot move progressively through the atomic structure, so that if by any means (such as rubbing) we cause an excess or dearth of electrons in any region, the resulting charge will remain localized in that region. We can no longer say that there can be no component of field intensity tangential to the surface

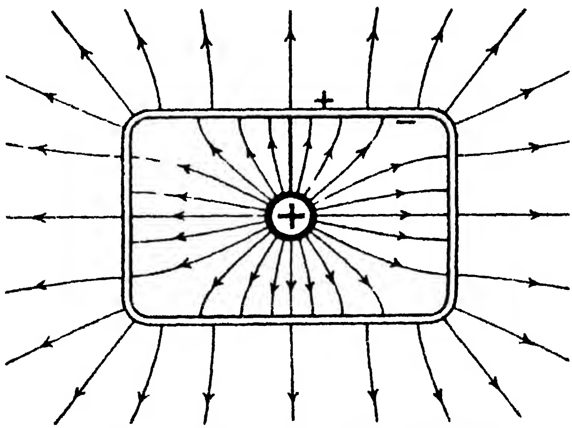


Fig 4 Charged body inside metal vessel

Again, when an insulating body is placed in an electric field, free charges no longer redistribute themselves in such a way as to make the net field inside the body zero Consequently there will be an electric field *inside* the insulator. In the majority of cases, however, we do find evidence of induced charges on the surface of the body, so that some sort of redistribution of electronic charges does take place.

Suppose we now cut an insulator into two pieces while it is situated in an electric field, by means of a cut in a plane perpendicular to the field. We then test each piece for a total charge, and find none. If we repeat this experiment with a

conducting body, however, we find that one half possesses an aggregate negative charge, and the other half an equal positive charge, showing that, when placed in the electric field, free charges were pulled completely over to the surface. The result in the case of an insulator can be explained by the hypothesis that individual atoms or molecules are *stretched* by the field in such a way that their component charges are pulled slightly away from their normal positions. Fig. 5 (A) gives a rough picture of a number of molecules (highly magnified) which have thus been stretched under the action of the external field. The left-hand surface of the body will possess a negative charge, the right-hand surface a positive charge, but any finite volume

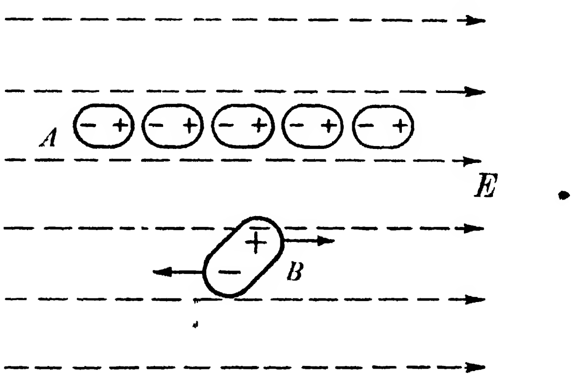


Fig. 5. A. Strained molecules. B. Dipole molecule

of the body as a whole will possess no aggregate charge, since in cutting up the insulator we cut between molecules, and not through them.

There is also experimental evidence for an additional cause of these induced surface charges: suppose that the normal state of a molecule is such that its component positive and negative charges have "centres of gravity", or "centres of charge", which are not coincident. It is then equivalent to an extremely small body with equal and opposite charges concentrated at different points, and will experience a turning moment, when placed in an electric field, such that the electric axis (the line joining the two points at which the opposite charges appear to be concentrated) will tend to align itself with the lines of force

of the field. When the insulator is not situated in a superposed field, the axes of the molecules are considered to be pointing in random directions, so that there will be equal numbers of positive and negative charges in any surface, which consequently shows no evidence of aggregate charge. When situated in an electric field, however, we imagine the individual molecules to rotate slightly, all the positive ends moving in the positive direction of the superposed lines of force, and all the negative ends moving in the opposite direction. Such a molecule, which is *polarized* in its normal state, is termed a *dipole*, and is illustrated roughly at B, in Fig. 5. The surfaces of the insulator will now show an induced charge on account of the rotation of the dipoles, but again no finite volume of the body will possess any aggregate charge.

When surface charges are induced on an insulator (or "dielectric") situated in an electric field, the dielectric is said to be *polarized*, and such polarization causes a diminution of the average field intensity inside the body. In the case of the conductor there is apparently no limit to the redistribution of the free or mobile charges, and the result is that the internal field intensity is reduced to zero. It is evident, then, that if internal charges move at all, they will move in such a way as to decrease the internal field, and if their motion is limited, as in an insulator, this internal intensity* is not reduced to zero.

Imagine now that the whole of the space occupied by the electric field due to two oppositely charged conductors is filled by a homogeneous insulating substance. The electric field intensity in this substance, or dielectric, due to the charged conductors will in most cases be less than it would be were they separated by free space or air. The relative reduction is found to be approximately constant for any given dielectric,

* The *local* field intensity inside the insulator is extremely variable, since it must be very great near an electron or a nucleus, while at intermediate points it may be zero. We surmount this difficulty by taking the *average* field intensity through the body, in the direction of the superposed field. This is equal to the total work involved in the travel of a unit point charge through the body, divided by the length of its journey.

and gives rise to the physical property known as the *dielectric constant*, or "relative permittivity", of an insulator, which is denoted by the letter K.

In most practical cases when a dielectric body is situated in the electric field of charged conductors, the field will be normal to the surface of the dielectric. This is easily seen to be true in the case of the usual types of capacitor. Suppose the space between the charged conductors and the dielectric to be evacuated, so that we can call it "free space" (air is almost as good), then when the field enters the dielectric normally its intensity falls to $1/K$ times its value in the external space, and this fact provides a simple physical explanation of the meaning of K.

Let the field intensity inside the dielectric be $\mathbf{E}$. Then this may be regarded as the resultant of two opposing field components, namely $\mathbf{E}_0$ due to the exterior charged bodies (the *polarizing* field), and the self-component $\mathbf{E}_d$, opposing $\mathbf{E}_0$, due to the molecular charges displaced by $\mathbf{E}_0$ in the dielectric. In vector notation, we then have the general relation

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d, \tag{10}$$

but if the polarizing field $\mathbf{E}_0$ is everywhere normal to the surface of the dielectric we also have

$$\mathbf{E} = \mathbf{E}_0/K, \quad \mathbf{E}_0 = K\mathbf{E},$$

so that

$$\mathbf{E}_d = -(K-1)\,\mathbf{E} = -(1-1/K)\,\mathbf{E}_0. \tag{10a}$$

In general, $\mathbf{E}_d$ is due to an equivalent surface and volume distribution of charge, but in the case we have considered, in which the dielectric is assumed to be homogeneous and isotropic, the volume charge does not arise and $\mathbf{E}_d$ can be attributed entirely to an equivalent surface distribution of charge.

If a *conducting* body is placed in the field $\mathbf{E}_0$, then the internal resultant field $\mathbf{E}$ is zero, so that $\mathbf{E}_d = -\mathbf{E}_0$.

The dielectric constant K is usually measured under con-

ditions where the electric field is alternating (e.g. by measuring the capacitance of a condenser, having the test material as dielectric: the definition of K in terms of capacitance is given in Section 9 below), and the value so obtained is not necessarily the same as that given by the above electro-static definition. The time taken for displaced atomic charges to take up positions of equilibrium, when there is a change in electric field intensity, is not indefinitely short, so that under a steady field a larger displacement of charges takes place than under conditions of alternating fields. Thus we may expect a decrease in K with increasing frequency, a decrease which is indeed observed, though in many cases the variation is not serious.

The dielectric constant for free space is clearly unity, and typical values for a few insulators are given below:*

Air (dry), at atmospheric pressure at 0° C	1·00058	Vinyl acetate	4
Water (20° C)	80	Ethylene	2·3
Amber	2·8	Mica	7
Porcelain	5·5–7	Crepe rubber	2·4
Soda glass	7·5	Vulcanized rubber	3·2
Lead glass	6·9	Paraffin wax	2·2
Fused quartz	3·8	Ebonite, pure	3
Oil-impregnated paper (1 kc)	2·3–3·6	Transformer oil	2·2

5. The theorem of Gauss.

The electro-static field, as defined by means of a point-charge, obeys the inverse-square law, but in practice the point-charge is non-existent and we have to deal with electro-static fields attending a finite distribution of elementary electronic charges. An extremely useful connecting link between the ideal and the practical case is found in the theorem of Gauss, which gives a simple relationship between the surface integral of the normal field intensity over any closed surface, and the total charge (however distributed) enclosed by the surface.

Consider any imaginary closed surface surrounding a point-charge q (Fig. 6) and let E be the field intensity at any point P of this surface. Let $E_n = E\cos\theta$ be the component of this intensity normal to the surface at P, where θ is the inclination of E to the normal Then the surface integral of E_n over the imaginary surface is

$$\iint E_n\,dA = \iint \frac{q}{4\pi K\epsilon_0 r^2}\cos\theta\,dA,$$

but
$$\iint \frac{\cos\theta\,dA}{r^2} = \iint \frac{dA_n}{r^2} = \iint dA_0 = 4\pi$$

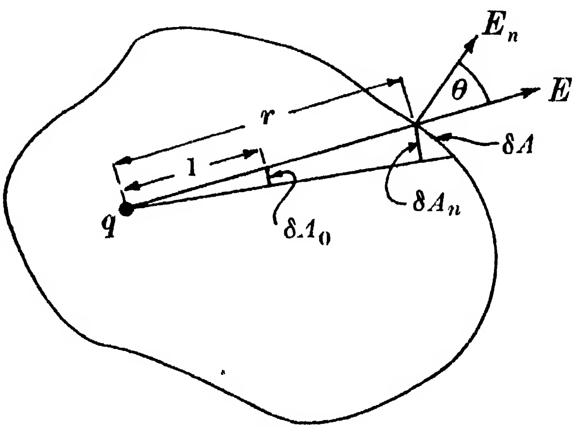


Fig 6

(In each case the integration is performed over the complete surface, in the figure δA_n is the element of the surface of a sphere of radius r cut off by the cone subtended by δA at the charge q while δA_0 is the value of δA_n when $r = 1$.)

Hence
$$\iint E_n\,dA = \frac{q}{K\epsilon_0}$$

or
$$K\epsilon_0 \iint E_n\,dA = q, \tag{11}$$

where K is the dielectric constant of the medium in which q is situated. Putting equation (11) in words:

> The charge enclosed by any closed surface in an electric field is obtained by integrating, over the surface, $K\epsilon_0$ times the normal field intensity.

It follows, by superposing the effects of any number of point-charges within the closed surface, that the theorem applies to any distribution of the total charge q.

Some applications of Gauss' theorem

A. *The field due to a uniformly charged sphere.*

Let E be the field intensity at a point outside the sphere, distant R from its centre. Let the sphere carry a uniformly distributed charge q (it is immaterial whether q is distributed over the surface, or throughout the volume, of the sphere). Then, by symmetry, E will be the normal field intensity at every point on the surface of an imaginary sphere of radius R. By Gauss

$$K\epsilon_0 \iint E_n \, dA = K\epsilon_0 E 4\pi R^2 = q,$$

so that

$$E = \frac{q}{4\pi K\epsilon_0 R^2} \tag{11a}$$

Thus the field intensity at any point exterior to a uniformly charged sphere *is the same as though the total charge were concentrated at its centre.*

B. *The field intensity at the surface of a charged conductor.*

In Fig. 7 the shaded portion represents a conductor situated in a dielectric of constant K, and which carries a surface charge of density σ per unit area.

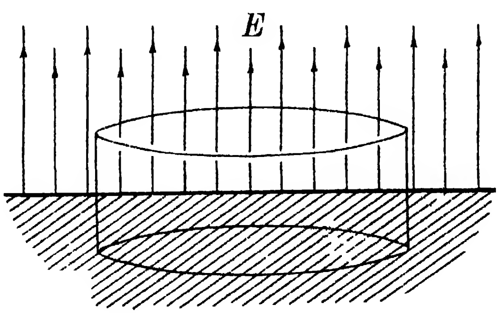


Fig 7

Imagine a closed surface formed by the walls of a cylinder whose plane ends have an area A and whose sides are perpendicular to the surface and therefore coincident with the

lines of force. Let one end of this cylinder be inside the conductor, while the other end is in the bounding dielectric.

In applying the theorem of Gauss, the only surface of the cylinder which has an electric field normal to it is the plane surface in the dielectric. Further, the total charge enclosed by the cylinder is σA. Hence

$$K\epsilon_0\iint E_n\,dA = K\epsilon_0 EA = \sigma A,$$

so that

$$E = \frac{\sigma}{K\epsilon_0}. \qquad (12)$$

C. *A tube of electric flux is terminated by equal charges.*

We have already seen from our definition of a tube of electric flux, that we should expect such a tube between two charged bodies to terminate on equal amounts of charge on each body. This is easily proved, where there is no change of dielectric (for which case see Section 6), by the theorem of Gauss

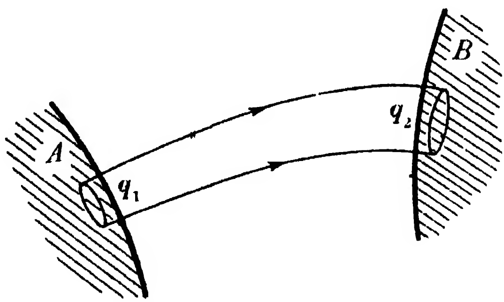


Fig. 8. Tube of displacement

Let A and B (Fig. 8) be two charged conductors between which an electric field exists. Consider a closed Gauss surface as shown, which is bounded in the space between A and B by lines of force, and whose ends are just inside the conductors. Then there is no normal component of field intensity on any part of this closed surface, and if the charge on the surface of A cut off by the Gauss surface is q_1 and that cut off on B is $-q_2$ (they must be of opposite sign) we have

$$K\epsilon_0\int E_n\,dA = 0 = q_1 - q_2,$$

so that

$$q_1 = q_2.$$

D. *Conditions at the boundary of two dielectrics in an electrostatic field.*

When an electric field passes from one medium to another, of different dielectric constant, it is distorted or refracted as shown in Fig. 9 (*a*).

Let the dielectric constants of the two media be K_1 and K_2, and let the horizontal line in Fig. 9 (*b*) represent their boundary surface. Let the field intensities in the two media, near the boundary, be E_1 and E_2, and let these be resolved into tangential components E_{t1} and E_{t2}, and normal components E_{n1} and E_{n2}.

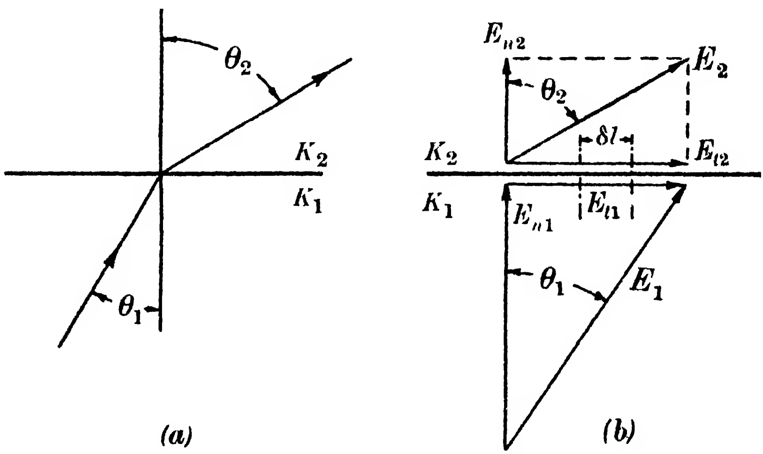


Fig. 9. Boundary of two dielectrics

(*a*) *Relation between the tangential components.* Consider E_{t1} and E_{t2} only. Then the potential difference δV between the dotted vertical lines, close to the boundary, is the same on both sides of the boundary. That is,

$$\delta V = E_{t1}\,\delta l = E_{t2}\,\delta l,$$

so that

$$E_{t1} = E_{t2}, \tag{13}$$

i.e. the tangential components of field intensity are equal.

(*b*) *Relation between the normal components.* Now suppose that only the normal components of field intensity are present, and imagine a cylindrical Gauss surface whose sides

are perpendicular to the boundary, and whose ends are parallel to it, one being situated in each medium. Assuming that the media contain no aggregate charge, the charge enclosed by this surface is zero, so that, by Gauss,

$$K_1 \epsilon_0 \iint E_{n1}\, dA - K_2 \epsilon_0 \iint E_{n2}\, dA = 0,$$

and hence $$K_1 E_{n1} = K_2 E_{n2}. \tag{14}$$

(*c*) *The general case.* The general case of Fig. 9 is obtained by superposing the results given by 1(13) and 1(14). Let θ_1 and θ_2 be the inclinations of E_1 and E_2 to the normal, then 1(13) and 1(14) become

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

and $$K_1 E_1 \cos\theta_1 = K_2 E_2 \cos\theta_2,$$

whence $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{K_1}{K_2}, \tag{15}$$

which is the required relation governing the conditions at the boundary.

We see from this result that when an electric field passes *normally* from free space into a dielectric of constant K, the field intensity is reduced to $1/K$ times its value in free space, but that when the field meets the surface of the dielectric at an angle it does not suffer such a large reduction. Putting $K_1 = 1$, and $K_2 = K$, it is easily seen from the above that a field of intensity E, on entering a dielectric of constant K at an angle of incidence θ, is reduced to a value given by

$$E_k = \frac{E}{K}\sqrt{K^2 - (K^2 - 1)\cos^2\theta}. \tag{16}$$

6. The electric flux-density, or displacement density.

We have already introduced the concept that from a charged particle q there emanates an *electric flux* ψ which is equal to q, and that at every point of the field there is a *flux-density*, or flux per unit normal area, D, which is a vector quantity and equal, in free space, to $\epsilon_0 E$ [1(5)]. The total flux from a given charge is independent of the dielectric constant of the surrounding medium so that, in general, the

flux passing through any closed surface surrounding a charge q must be equal to q. That is,

$$\iint D_n \, dA = q, \tag{17}$$

the left-hand side being the surface-integral of D over the closed surface. Comparing 1(17) with 1(11), (the Theorem of Gauss), we see that the two are identical if

$$\mathbf{D} = K\epsilon_0 \mathbf{E}. \tag{17a}$$

Provided the dielectric is homogeneous and isotropic, so that K is a simple numerical coefficient, we may accept 1(17a) as a general definition of electric flux-density. As examples of simple cases, we see from 1(11a) that the flux-density at a distance R from a point-charge q, or from the centre of a uniformly charged sphere, is given by

$$D = \frac{q}{4\pi R^2}, \tag{17b}$$

and that near the surface of a charged plane conductor the flux-density is

$$D = \sigma. \tag{17c}$$

The concept of electric flux-density is due to Clerk Maxwell, who based it on the hypothesis that electric and magnetic forces are transmitted by means of a universal physical medium, or aether, filling all space. In this medium he supposed that an electric force, which we denote by E, produces an actual strain or *displacement* of electricity. In the case, for instance, of a charged sphere he concluded that: "The displacement outwards through any spherical surface concentric with the sphere is equal to the charge on the sphere",* and in general he postulated: "That whatever electricity may be, and whatever we may understand by the movement of electricity, the phenomenon which we have called electric displacement is a movement of electricity in the same sense as the transference of a definite quantity of electricity through a wire is a movement of electricity, the only difference being that in the

* *A Treatise on Electricity and Magnetism*, J. C. Maxwell (3rd. ed.) Vol. 1, p. 67.

dielectric there is a force which we have called electric elasticity which acts against the electric displacement, and forces the electricity back when the electromotive force is removed."* This is certainly a reasonable interpretation of what happens when a material dielectric is polarized, but Maxwell's particular hypothesis was that it *also applies to free space*, and he further supposed that "The variations of electric displacement evidently constitute electric currents".† All this may be summarized in the hypothesis that even in free space, a changing electric field is attended by a magnetic field, and this is the particular contribution of Maxwell which led to his theory of electromagnetic waves.

Mathematically, the vector field-measure D is analogous to the magnetic flux-density, B, and is therefore more usually known as the *electric flux-density*, rather than by the older names *displacement density* or *induction*.

Maxwell's hypothesis of a material aether has long since been discarded, but the quantity D remains an essential part of classical electromagnetic theory which is still based on his mathematical equations We no longer think of an actual movement of electricity in a changing free-space electric field, but we still speak of dD/dt as a "displacement current density", which if not physically consistent with the viewpoint of modern physics is at least a suitable reminder of Maxwell's genius.

Conditions at the boundary of two dielectrics, in terms of D and E. At the boundary of two dielectrics the electric field is refracted in such a way that the *tangential* components of E, the field intensity, are unaltered [see 1(13)], while from 1(14) and 1(17a) it follows that the *normal* components of electric flux-density are unchanged:

$$D_{n1} = D_{n2}. \tag{18}$$

The electric flux is continuous through a dielectric boundary. It follows at once from 1(18) that electric flux is continuous through a dielectric boundary if it enters the boundary normally, and it may be easily seen that this is also true if the field meets the boundary obliquely. Let a hori-

* *loc. cit.*, p. 69. † *loc. cit.*, p. 65.

zontal line represent the boundary surface of two dielectrics of constants K_1 and K_2, and let a unit tube of flux pass from one medium into the other, meeting the surface at angles θ_1 and θ_2 to the normal respectively. Let A_1 and A_2 be the cross-sectional areas of the tube on each side of the boundary. Then if D_1 and D_2 are the respective flux-densities the total flux through the surface A_1 is

$$\psi_1 = A_1 D_1 = A D_1 \cos\theta_1 = A D_{n1},$$

and the total flux through the surface A_2 is

$$\psi_2 = A D_{n2},$$

so that it follows, from 1(18), that

$$\psi_1 = \psi_2.$$

7. The polarization.

The electric flux-density, $\mathbf{D}$, given by 1(17a) for a material dielectric, may be split up into two parts:

$$\mathbf{D} = K\epsilon_0 \mathbf{E} = \epsilon_0 \mathbf{E} + (K-1)\epsilon_0 \mathbf{E}. \qquad (18a)$$

The first part, $\epsilon_0 \mathbf{E}$, is the flux-density of the resultant field $\mathbf{E}$ in the free space between the material particles in the dielectric, while the second part can be interpreted as an actual displacement or shift of electric charges in the dielectric structure.

Fig. 10 illustrates the simple case of a plate of dielectric inserted between the charged conducting plates of a parallel-plate capacitor, leaving air spaces on each side. The relation between the field intensity in the air spaces and that in the dielectric is $\mathbf{E} = \mathbf{E}_0/K$, and the reduction of the internal field is due to the self-component of field, $\mathbf{E}_d$, given by 1(10a):

$$\mathbf{E}_d = -(K-1)\mathbf{E} = -(1-1/K)\mathbf{E}_0.$$

This self-component must arise from an induced surface charge on each face of the dielectric plate, σ_{p+} and σ_{p-} as shown, and by 1(17c) this must be numerically equal to the free-space flux-density corresponding to $\mathbf{E}_d$. That is, we have, numerically, from 1(10a):

$$\sigma_p = (K-1)\epsilon_0 E, \qquad (18b)$$

which is the last term of 1(18a). This term represents an actual displacement of charges throughout the dielectric,

giving rise to the surface charges σ_{p+} and σ_{p-}. The dielectric is *polarized*, and if the thickness of the plate is t it follows that the surface charges on unit area constitute an *electric dipole* whose moment is σt, directed as a vector from the negative to the positive surface charge. The *dipole moment per unit volume*

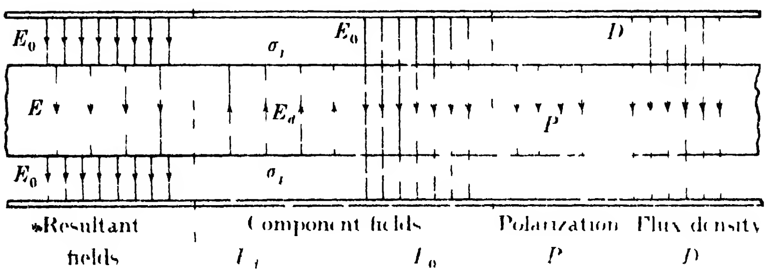


Fig 10

is termed the *polarization*, **P**, and is clearly equal, in this simple case, to σ_p. In general the relation is

$$\sigma_p = P_n, \tag{18c}$$

where P_n is the normal component of **P**, reckoned positive when directed outwards from the dielectric surface. We can now write 1(18*a*) in the form

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{18d}$$

a relation which is generally valid. The ratio P/E is called the *dielectric susceptibility*.

8. Capacitance.

Consider two uncharged conductors separated by an insulating medium. The transfer of electrons from one to the other will give to the conductors equal and opposite charges, and at the same time a potential difference will appear

between them, which will be proportional to the charge transferred. Thus we may put

$$Q = CV, \tag{19}$$

where Q is the charge on each conductor, V is the potential difference between the conductors, and C is a constant, for the particular arrangement of conductors considered, called the *capacitance* (or capacity) of the conductors, which are said to form a *capacitor.*

The two conductors are called the *plates* of the capacitor, and Q, the charge on *either* plate, is called the charge of the condenser. Putting V equal to unity in 1(19), we obtain the following definition of capacitance·

> The capacitance of a capacitor is equal numerically to the charge necessary to produce a potential difference of unity between the plates

If Q is in coulombs, and V in volts, then C is in *farads.* A more convenient unit, in practice, is the *microfarad* (1μF.), which is equal to 10^{-6} farad.

9. The energy stored in an electric field.

In charging a capacitor, work must be done in setting up the p.d. between the plates, and this work, or energy, may be recovered when the capacitor is discharged. Thus a charged capacitor (i e. any system of charged conductors between which there is an electric field) possesses potential energy, and owing to the mathematical analogy with a strained elastic body, it is usually stated that this energy is "stored" in the insulating medium The storage of energy, as strain energy, in a material dielectric which is situated in an electric field is readily accepted since an actual physical "strain" is considered to exist, by the slight displacement of atomic charges under the action of the field. When the dielectric is free space, however, we have no knowledge of the "mechanism" by which energy is "stored", and when we calculate the energy stored, per unit volume, in such a field we do so only because the theory leads to results which are consistent with experiment.

A. *The energy stored in the field of a charged capacitor.*

Let the plates initially be at the same potential, so that there is no electric field between them. During the process of transferring charges from one plate to the other, let the charge on the capacitor at some instant be q and the corresponding potential difference v. Now let a small charge δq be transferred, then, by the definition of potential difference, the energy expended in this operation is

$$\delta W = v\,\delta q$$

and the potential difference will increase by δv, where

$$\delta v = \frac{\delta q}{C}, \quad \text{from } 1(19)$$

whence $$\delta W = Cv\,\delta v.$$

The total work expended in charging the capacitor to a potential difference V and charge Q (i.e. the energy stored in the electric field between the plates) is therefore

$$W = C\int_0^V v\,dv = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV. \qquad (20)$$

B. *The energy stored per unit volume of a dielectric.*

Now take a portion of the field of a parallel-plate capacitor in which the field is uniform and normal to the surface of the plates (a region near the centre of the plates, where the distance between the plates is small compared with their linear dimensions, will conform with this condition). Let the area of each plate bounding this portion of the field be A, and the corresponding charge (on A) be Q. Let the distance between the parallel plates be d.

Then $\psi = Q =$ total displacement in the field bounded by the areas A. The displacement density is

$$D = \frac{\psi}{A} = K\epsilon_0 E$$

or $$Q = \psi = AK\epsilon_0 E.$$

The potential difference between the plates is

$$V = Ed,$$

so that the energy stored in the region considered is, by 1(20),

$$W' = \tfrac{1}{2}QV = \frac{AdK\epsilon_0 E^2}{2}.$$

Since Ad is the volume of the region, the energy stored per unit volume of the field is

$$\left.\begin{aligned} W &= \frac{K\epsilon_0 E^2}{2} \\ &= \frac{D^2}{2K\epsilon_0} \\ &= \frac{ED}{2} \end{aligned}\right\} . \tag{21}$$

This is mathematically analogous to $\frac{1}{2}$(stress) × (strain), which gives the energy stored per unit volume of a stressed elastic body.

10. The effect of the dielectric constant, K, on capacitance.

For a given capacitor and a fixed value of the p.d., V, consider the two cases where the dielectric between the plates is

(A) Free space (air is approximately equivalent), for which $K = 1$.

(B) A dielectric of constant K

The field intensity, E, is the same in both cases, and by comparing equation 1(20) with the first of the three forms of 1(21), it is seen that the capacitance C is directly proportional to the dielectric constant K. This is a result we should expect from a consideration of the definition of capacitance, together with the reduction of the potential gradient for a given displacement when the dielectric constant increases from 1 to K.

We therefore have a very useful definition of K:

SECOND DEFINITION OF K

The dielectric constant, K, of a dielectric is equal to the ratio of the capacitance of a given capacitor when its plates are separated by the given dielectric, to the capacitance when the plates are separated by a vacuum.

11. The mechanical force acting on the charged surface of a conductor.

We shall first suppose that the charged conductor, in a dielectric equivalent to air or free space, is disconnected from the supply so that its charge remains constant and no energy can be supplied from an external source.

Let the surface (Fig. 11) have a charge of σ per unit area. Then the field intensity close to the surface is, by 1(12),

$$E = \frac{\sigma}{\epsilon_0}.$$

The energy stored in this field, per unit volume, is

$$\frac{ED}{2} = \frac{\sigma^2}{2\epsilon_0}$$

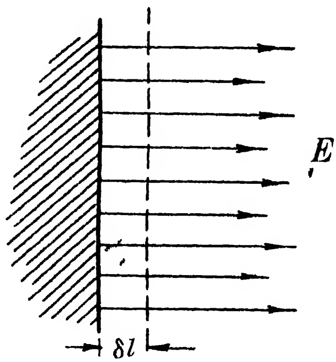


Fig 11

Now suppose that the charged body moves in such a way that the surface moves a short distance δl in the direction of the lines of force, under the attraction of the charges on the other boundary of the field. If δl is very small, $\mathcal{E}$ can be assumed to remain constant.

Then if F is the force experienced by the surface per unit area, the work done is $F\,\delta l$, and this must be equal to the energy released from a volume of $(1 \times \delta l) = \delta l$ (which has been swept through by the unit area of surface) in which the field intensity E has disappeared.

Thus $\quad F\,\delta l = \dfrac{\sigma^2}{2\epsilon_0}\,\delta l \quad$ and $\quad F = \dfrac{\sigma^2}{2\epsilon_0}$ per unit area. $\qquad$ (22)

The force between the plates of a capacitor. It is of interest to examine a similar problem by means of a more general analysis of energy relations. In Fig. 11, suppose the shaded conductor to represent the left-hand plate of a parallel-plate capacitor, the distance between the two plates being x. Suppose the capacitor to be connected to a d.c. source of p.d. V, and that the left-hand plate moves a short distance δx towards the other plate, so that the increment in x is $-\delta x$. Let the charge on the capacitor be Q and its capacitance C, then in general the small movement will be accompanied by increments δV, δQ, and δC in V, Q and C. The energy supplied from the source will be $V\,\delta Q$, and this must equal the sum of the mechanical work done and the increment in the field energy. So if F is the force of attraction between the plates, we have

$$V\,\delta Q = F(-\delta x) + \tfrac{1}{2}(V\,\delta Q + Q\,\delta V). \tag{22a}$$

so that

$$F = \frac{1}{2}\left(Q\frac{dV}{dx} - V\frac{dQ}{dx}\right). \tag{22b}$$

It is more convenient to express this in terms of V and C, and since $Q = CV$ we have

$$\frac{dQ}{dx} = C\frac{dV}{dx} + V\frac{dC}{dx},$$

and substituting this in 1(22a):

$$F = -\tfrac{1}{2}V^2\frac{dC}{dx}. \tag{22c}$$

By means of 1(23) below, it may easily be confirmed that this result is consistent with 1(22).

Consider 1(22a) in each of two cases. First suppose that the p.d. applied to the capacitor is maintained constant so that $\delta V = 0$. It then follows that the energy supplied, $V\,\delta Q$, must be equally divided between mechanical work, $F(-\delta x)$, and the increment in field energy $\tfrac{1}{2}V\,\delta Q$. For a given amount of mechanical work, therefore, twice this amount must be supplied from the source. On the other hand, if the capacitor is disconnected from the supply so that $\delta Q = 0$, it follows from

1(22*a*) that the mechanical work must be provided by a decrease in the field energy. In both cases, however, the force is given by 1(22) or by 1(22*c*).

12. The capacitance of certain capacitors.

A. *The capacitance of a parallel-plate capacitor.*

The configuration of the electric field between the plates of a parallel-plate capacitor is sketched in Fig. 12. Except in

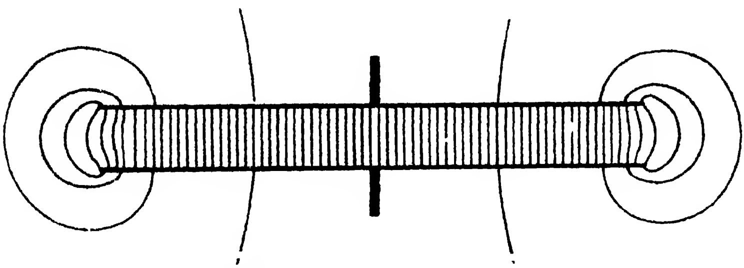


Fig 12 Parallel plate capacitor

the region near the edges of the plates, the field is uniform and normal to the plates In the following calculation we assume that the field has this uniform distribution throughout, a condition which can be obtained in practice by the use of "guard rings", which are plates surrounding the actual capacitor plates, coplanar with them, and separated from them by a small gap, if the guard rings are kept at the same potential as the capacitor plates which they surround, any serious non-uniformity of field distribution is limited to the region near the outside edges of the guard plates, and the field in the capacitor is uniform (see Fig 13) In the absence of guard

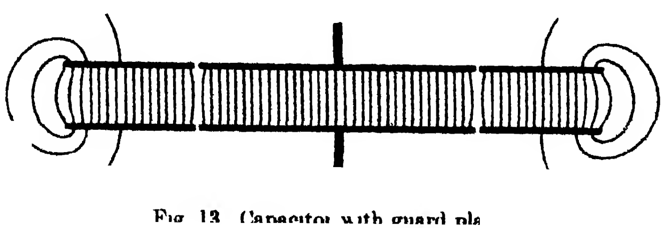


Fig 13 Capacitor with guard pla

rings, the following theory gives a result with no appreciable error so long as the plates are large compared with their distance apart.

Let the distance between the plates be d. Then the capacitance is equal to Q/V where V is the p.d. between the plates. Since $V = Ed$ and $Q = DA$, where A is the area of each plate, and $D = K\epsilon_0 E$, we have

$$C = \frac{Q}{V} = \frac{DA}{Ed} = \frac{K\epsilon_0 A}{d} \tag{23}$$

B. *The capacitance of a concentric-cylinder capacitor.*

The two plates of the capacitor are concentric cylinders of radius a and b, and of equal length If the cylinders are prolonged by guard cylinders (of the same radii, separated from the capacitor cylinders by a short gap, and kept at the same potential as the plates they "guard") the field intensity between the plates can be taken as radial and uniform over the axial length of the plates

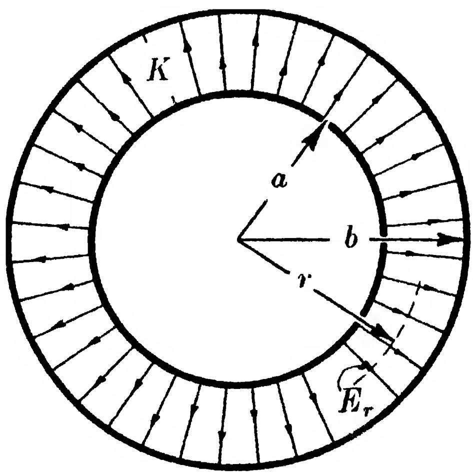


Fig. 14*a*. Concentric-cylinder capacitor

Let E_r be the potential gradient (Fig. 14*a*) at any point between the plates and at a distance r from their common axis. Apply Gauss' theorem to a closed cylindrical surface of radius r

and unit length: then the only part of this cylindrical surface which has a field intensity normal to the surface will be the cylindrical walls, so that we have

$$\iint K\epsilon_0 E_n \, dA = K\epsilon_0 E_r \, 2\pi r = Q.$$

where Q is the charge per unit axial length of the conductor. Whence

$$E_r = \frac{Q}{2\pi K\epsilon_0 r}$$

and the p.d. between the cylinders is

$$V = \int_a^b E_r \, dr = \frac{Q}{2\pi K\epsilon_0} \log_e \frac{b}{a}.$$

Hence $$C = \frac{Q}{V} = \frac{2\pi K\epsilon_0}{\log_e \frac{b}{a}}, \text{ per unit of axial length,}$$

$$= \frac{2\pi K\epsilon_0 L}{\log_e \frac{b}{a}}, \qquad (24)$$

where L is the axial length of the capacitor.

(*c*) *The capacitance of a concentric-sphere capacitor*

Let Fig. 14*a* now represent a section through the centre of two concentric spheres. The electric field between the inner sphere and the outer hollow sphere is everywhere radial, and is uniform over any imaginary concentric sphere of radius r. Let a and b be the outer and inner radii of the inner and outer spheres respectively, and let $\mathscr{E}_r$ be the potential gradient at any point on the surface of an imaginary sphere of radius r, where $b > r > a$.

Now the field inside the outer sphere, due to its charge, is zero, so that $\mathscr{E}_r$ is due entirely to the charge Q on the inner sphere. From Section 5, A, we therefore have

$$E_r = \frac{Q}{4\pi K\epsilon_0 r^2},$$

so that the p.d. between the spheres is

$$V=\int_a^b E_r\,dr=\frac{Q}{4\pi K\epsilon_0}\frac{(b-a)}{ab};$$

and $$C=\frac{Q}{V}=4\pi K\epsilon_0\frac{ab}{(b-a)}. \tag{25}$$

Corollary. The capacitance of an isolated sphere of radius a is obtained from 1(25) by making b infinite. Hence

$$C=4\pi K\epsilon_0 a. \tag{26}$$

Thus the capacitance of an isolated sphere is directly proportional to its radius

D. *The total capacitance of a bank of capacitors in parallel.*

Let a bank of capacitors of capacitance C_1, C_2, .., C_n be connected in parallel as in Fig. 14*b*. Then if they are charged to a p d. of unity the total charge stored in the bank is

$$Q=C_1+C_2+\ldots+C_n.$$

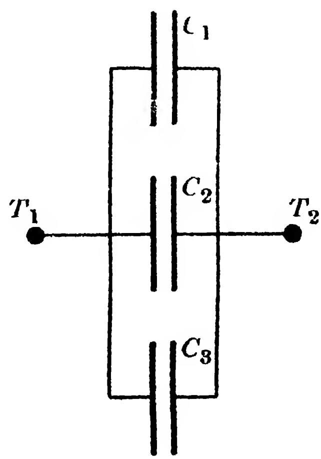


Fig. 14*b*. Capacitors in parallel

But this is the equivalent capacitance of the bank. Therefore

$$C=C_1+C_2+\ldots+C_n. \tag{27}$$

E. *The total capacitance of a bank of capacitors in series.*

Suppose the bank is charged to a p.d., V, with polarity as indicated in Fig. 14c. Let the positive charge on the left-hand

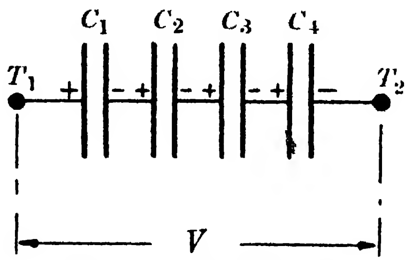


Fig. 14c Capacitors in series

plate of C_1 be $+q$, then the charge on the right-hand plate is $-q$, leaving a charge $+q$ on the left-hand plate of C_2. Consequently each capacitor of the bank carries an equal charge, which (i e. q) must pass the terminals T_1 and T_2 when the bank is charged. Thus the bank as a whole carries a charge q. Hence we have, if V_1, V_2, etc are the p d.'s across the component capacitors.

$$V = V_1 + V_2 + \dots + V_n = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}\right).$$

Now the equivalent capacitance C is equal to q when $V = 1$ so that

$$C = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dots + \dfrac{1}{C_n}}. \qquad (28)$$

PART II

THE ELECTRIC CURRENT

1. Introduction.

Up to about the end of the eighteenth century, the experimental knowledge of electricity in motion, as distinguished from electro-statics, was limited to the transient discharges of electro-statically charged bodies. The division of material substances into insulators and conductors had been discovered by Stephen Gray in 1729, and the idea of a "motion"

of electricity through conductors probably originated as a logical consequence of this discovery. Knowledge of sustained currents began with Galvani in 1786, for which reason the phenomena were long classed under the name of "galvanism". The controversy between Galvani and Volta, as to the source of the phenomena, is well known, and culminated in the invention of Volta's "pile", or primary battery, about 1799.

Volta's invention was of outstanding importance, both practically and theoretically. For the first time there was now provided a source of *continuous* currents for experimenters, and it was also shown conclusively that the effect had its origin, not in animal tissue as Galvani had maintained, but in the contact of dissimilar conductors. There quickly followed (in 1800) the discovery of the decomposition of water by "galvanism", from which arose an increasing knowledge of the laws of electrolysis.

During this period the evidences of a "galvanic" current were limited to such effects as the convulsion of animal tissue, the heating of a thin wire, decomposition of electrolytes, and the production of a spark No definite connection with magnetism had been found, although the disturbing effect of lightning upon magnetic compass needles had suggested such a possibility.

When Oersted, in the winter of 1819–20, discovered that a galvanic current affected a compass needle, the modern science of electro-magnetism was born. The discovery not only stimulated an intense activity in research which resulted in the crowning achievement of Faraday in 1831, but provided a new test for an electric current which superseded the older and more complicated methods. Moreover, through the brilliant work of Ampère, and the application of the theory of magneto-statics, this property of a current (of being attended by a magnetic field) very quickly put "galvanism" on a sound quantitative basis.

The fundamental physical meaning of an electric current now became "something which produces a magnetic field", a physical basis which could not be improved upon until the discovery of the electron, and the formulation of the electronic

theory of conduction, long after the classical theoretical work of Clerk Maxwell.

Maxwell adopted the electro-magnetic system of units suggested by Weber, in which electric current is defined in terms of the magnetic field it produces. In criticizing this definition, we must remember that, in Maxwell's day, a magnetic field was looked upon as something basically fundamental, and the definition was not only theoretically sound but physically logical. At the present day, however, the definition is not logically consistent with our knowledge, for it leads to the definition of quantity or charge of electricity as something which moves when current flows. That is, taking the unit magnetic pole as the fundamental starting-point, we are led, through the magnetic field and electric current, to electric charge as the most hypothetical concept, whereas we now regard it (through our acceptance of the fundamental nature of the electron) as a basic reality. From our point of view, then, the magnetic pole is the most hypothetical of concepts, and it is illogical to base definitions of current and charge upon it.

It remained for H. A. Rowland, in 1876,* to show experimentally that the electric charge of electro-statics could be identified with the hypothetical charge of the electro-magnetic system. On the basis of such an assumption several theories, such as that of Weber,† had already been postulated, in which electric charges were regarded as fundamental, and in which electro-magnetic phenomena were explained in terms of the forces between charges, these forces depending upon the positions, velocities, and accelerations of the charges. The trend of modern physics, since the discovery of the electron, has been to substantiate the fundamental nature of electric charge, and to make more and more hypothetical the field concepts of Faraday and Maxwell. At the present time electrical engineers are familiar with the electric current in its simplest form, that of a stream of electrons in a high-vacuum chamber such as a cathode-ray oscillograph.

* *Ann. d. Phys.* CLVIII (1876), p. 487.

† W. Weber, *Ann. d. Phys.* LXXIII (1848), p. 193.

2. Conduction current.

By a conduction current we mean the flow of electric charges under the action of an electric field. The phenomenon is always attended by energy transformations, and by far the commonest example is the current in a closed metallic circuit, though currents in liquids and gases, and the motion of electrons in vacuum tubes, are covered by the term.

The current flowing in a conducting circuit whose section at any point has definite boundaries (for example, a copper wire) is defined as the rate at which electric charges pass any given section of the conductor. Unit current thus flows when unit charge passes a given section in unit time (the practical unit being the *ampere*, which is a flow of one coulomb past a section in one second). By this statement we define the current through any given section of the circuit, but this is not necessarily the same for all sections. In cases of steady and quasi-steady (slowly changing) currents, however, the conduction current in a closed conducting circuit which is perfectly insulated is found to be single-valued, or of the same magnitude at all parts of the circuit. In this it is analogous to the steady flow of water in a pipe, and just as a knowledge of the rate of flow, in cubic feet per second, gives no information as to the *velocity* of the water, so the magnitude of a current gives no information about the velocity of the moving charges. This depends also upon the magnitude of the moving charge, per unit length of the conductor.

Consider a length L (Fig. 15) of a conducting wire of uniform

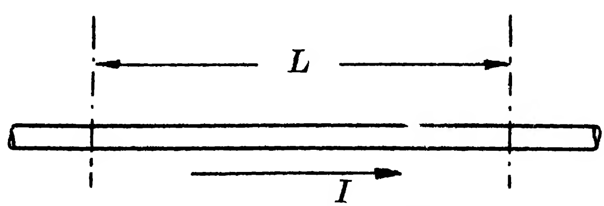


Fig. 15. Conduction current

section, in which a steady current of I units is flowing. We consider this current to be due to a continual drift or mass motion of a large number or free electrons, uniformly dis-

tributed throughout the conductor, under the action of an electric field. Let Q denote the charge of these electrons per unit volume of the conductor, and let v be the mean velocity of drift of the electrons along the wire. Let A be the constant cross-sectional area. Then I is equal to the number of units of charge passing through any cross-section in unit time, or

$$I = QAv. \tag{29}$$

The *current-density*, or the current per unit area of cross-section, is

$$J = \frac{I}{A} = Qv. \tag{30}$$

In general, the current-density J gives more definite information than the total current I, since the former gives complete information of conditions *at a point*. Only in special cases (such as steady currents in long narrow conductors) is the current-density uniform across a given cross-section, and in the general case (for example, high-frequency alternating currents, or the current between electrodes immersed in a large tank of conducting liquid) a knowledge of J at every point is required for a complete statement of the facts.

The ampere in terms of electrons. As an example of 1(30) we may take the interesting problem of the probable velocity of drift of electrons in a copper wire when a current of normal density flows. Since the charge of one electron is $1{\cdot}602 \times 10^{-19}$ coulomb, it follows that a current of one ampere, due to moving electrons alone, must involve the passage of $10^{19}/1{\cdot}602$ or $6{\cdot}24 \times 10^{18}$ electrons past any section of the circuit in one second. The mean velocity of drift of these electrons is, however, very small.

Let J be the current-density in amperes per sq. cm.

$$= Qv.$$

The quantity Q is not known with any degree of certainty, but there is reason to suppose that it is of the order of 5×10^{22} electrons, or roughly 8000 coulombs, per cubic cm. Hence

$$v \doteqdot \frac{J}{8000} \text{ cm per sec.}$$

Thus with a current density of 2000 amps per sq. in. (310 amps

per sq. cm.), the mean velocity of the electron drift is only about 0·04 cm. per sec.

3. The power involved in the flow of conduction current.

Under normal conditions, work is always involved in the passage of current through a conductor. This may be merely the work necessary to keep the electrons drifting through the atomic structure (i.e. against the *resistance* of the wire), in which case the energy expended is liberated as heat, or conversion of energy by electro-magnetic action may also be taking place.

Let there be a potential difference V between the terminals of a circuit through which flows a current of I units. Then by the definition of p.d. the work involved in taking Q coulombs through the apparatus is equal to VQ. Now the current I is equal to the charge transferred from terminal to terminal per second (it is not necessary that this charge pass completely through the conductor), so that the work per second (power) is equal to VI, i.e.

$$P = VI. \tag{31}$$

(In practical units, watts = volts × amperes.)

4. The conventional direction of current flow.

The conventional direction of fall of electro-static potential has been defined as that direction in which positive charges would move under the action of the field. That is, a positive charge is at a *higher* potential than a negative charge. This is in keeping with the usual numerical significance of plus and minus signs. A similar convention has been adopted for the direction of current flow: *the positive direction of current flow is the direction of motion of positive charges.* In certain cases of conduction, we know that both positive and negative charges take part in the current flow, and clearly their motion is in opposite directions. In the case of current flow in metals, however, and also in high-vacuum tubes, the current seems to consist wholly of a movement of electrons (negative charges) so that the real material motion in such cases is *opposite* to the conventional direction of current.

5. Resistance: Ohm's Law.

When an electric current flows in a conductor at normal temperatures the conductor becomes heated. It follows that electrical energy is being converted into heat energy, so that there must be a fall of potential along the conductor in the direction of current flow. For a given metallic conductor at a constant temperature, in which there is no induced e.m.f., it is found that the p.d. from end to end is proportional to the magnitude of the current, or

$$V = IR, \tag{32}$$

where R is called the *resistance* of the conductor. This important relation is known as *Ohm's Law*, and the practical unit of resistance, corresponding to the volt and the ampere, is the *ohm*.

The power necessary to maintain the current I in a resistance R is

$$P = VI = \frac{V^2}{R} = I^2 R. \tag{33}$$

6. Resistivity and conductivity.

For conductors of uniform section and of the same homogeneous material, and at the same temperature, it is found that the resistance R is directly proportional to the length, and inversely proportional to the area of cross-section, of the conductor. That is

$$R = \frac{\rho L}{A}, \tag{34}$$

where L is the length of the conductor, A is its uniform cross-sectional area, and ρ is a constant, for a given material at a given temperature, called the *resistivity* or the *specific resistance* of the material.

For the case of a current in a uniform conductor, the field intensity along the conductor, E, is uniform throughout its length, so that $V = EL$. Hence the current-density

$$\left.\begin{aligned} J &= \frac{I}{A} = \frac{V}{RA} = \frac{V}{\rho L} = \frac{E}{\rho} \\ \text{or}\quad E &= \rho J. \end{aligned}\right\} \tag{35}$$

This relation is in fact generally valid, and is the basic law of

conduction for a stationary conductor. The reciprocal of ρ is called the *conductivity* of the material, and is denoted by γ. Thus

$$J = \gamma E, \tag{35a}$$

and the reciprocal of the resistance R is called the *conductance*, G, of the conductor, so that

$$G = \frac{1}{R} = \frac{\gamma A}{L}. \tag{35b}$$

7. Variation of resistance with temperature.

It is well known that, for a considerable temperature range, the resistivity of metals varies almost linearly with the temperature. Consequently we may put

$$R_2 = R_1\{1 + \alpha_1(t_2 - t_1)\}, \tag{36}$$

where R_1 is the resistance of a conductor at temperature t_1 and R_2 is its resistance at temperature t_2, while α_1 is called the *temperature coefficient of resistance* of the material at temperature t_1.

For pure metals in the solid state, the temperature coefficient at 0° C. (α_0) is roughly equal to 0·004. If the resistance dropped linearly to zero at 0° Absolute (– 273° C.), α_0 would be equal to 0·00366, a figure which is very closely approximated for gold and platinum.

If the temperature coefficient at one temperature, t_1, is known, it is a simple matter to find its value at some other temperature t_2, provided that t_1 and t_2 are within the range of linear variation.

Let $\qquad \alpha_1 =$ temperature coeff. at t_1,

$\qquad \alpha_2 =$ temperature coeff. at t_2,

then $\qquad R_2 = R_1\{1 + \alpha_1(t_2 - t_1)\}$

and $\qquad R_1 = R_2\{1 + \alpha_2(t_1 - t_2)\}$,

whence, by eliminating R_1 and R_2,

$$\alpha_2 = \frac{\alpha_1}{1 + \alpha_1(t_2 - t_1)}. \tag{37}$$

8. Electromotive force (e.m.f.).

We saw in Section 3 above that the power involved, when a current I flows through a circuit having a p.d. between its terminals equal to V, is given by $P=VI$.

Now a p.d., V, is strictly a measure of the difference of electrostatic potential between two points (e.g. the terminals), and when such points are joined by a conducting path, there must clearly be some active *cause* of the separation of charges in the conductor which gives rise to the p.d.

If the conductor is a solid metal, we have good reason to believe that the only movable charges are free electrons, and in order to cause a p.d. between the ends of a metal wire it is necessary that the distribution of free electrons should be disturbed This will happen if the conductor is suitably situated in an electric field, since owing to this field, whatever its origin, the free electrons will experience forces, and the redistribution of the electrons proceeds until the electro-static field of the resulting distributions of charge exactly cancels the superposed field which *caused* the redistribution, provided that the conductor does not form part of a closed circuit. If the circuit is closed, and current is flowing, then a net electric field must exist inside the conductor of value $E=\rho J$ (equation 1(35)), and to this extent the opposing fields will not cancel each other,

The distribution of free electrons in the metal wire may also be upset by a difference in temperature between its ends. If we regard the free electrons, which have mass in addition to electric charge, to behave roughly in the manner of the molecules of a gas, an increase in temperature at one end of the wire will increase the electron-gas *pressure* (i.e. the random velocities of the electrons). Consequently, in an attempt to equalize this pressure throughout the conductor, some electrons will move towards the cooler terminal, and again a p.d. will result.

In primary or secondary cells, a redistribution of charges takes place due to the inter-atomic forces of chemical action, and this redistribution causes the terminal p.d. Again, when different metals are in contact, differences in the binding energy of the outer electrons result in a shifting of electrons from one

metal to the other, and so there results a difference in potential between the two metals.

Whatever may be the fundamental nature of the *cause* of this separation of charges, we term it *electromotive force* (e.m.f.). Since, in general, the result of the e.m.f. is to cause an electrostatic field (of the displaced charges) having a p.d., V, it is reasonable to measure e.m.f. in the same units as potential difference.

It should be noted that e.m.f. cannot be directly measured. In a closed circuit it causes a current, and it can be calculated from a knowledge of the current and the resistance of the circuit, while in an open circuit it causes a potential difference between the terminals which can be measured. This open-circuit p.d. is, in fact, accepted as a measure of the e m.f. which causes it, so that we are led to define e.m.f. as the line-integral of the "force per unit charge" which causes the potential difference.

Suppose a stationary open-circuited conductor, whose terminals are A and B, is situated in an electric field of intensity $\mathbf{E}_i$ which *excludes* the field of charges displaced by $\mathbf{E}_i$ and which has a component directed from A to B. Then, as explained above, the charges displaced in the conductor by $\mathbf{E}_i$ give rise to an electrostatic component of field $\mathbf{E}_d$ which reduces the resultant field to zero so that $\mathbf{E}_d = -\mathbf{E}_i$. The field component $\mathbf{E}_d$ will therefore be directed from B to A and

$$\int_A^B \mathbf{E}_i \cdot d\mathbf{l} = \int_B^A \mathbf{E}_d \cdot d\mathbf{l}.$$

The left-hand term gives the e.m.f. from A to B, and the right-hand term the fall of potential from B to A. We therefore define the e.m.f. in an open-circuited stationary conductor as the line-integral of $\mathbf{E}_i$ along the conductor, or

$$e_{AB} = \int_A^B \mathbf{E}_i \cdot d\mathbf{l}. \tag{37a}$$

Electromagnetic induction of currents requires a resultant e.m.f. around a closed circuit, i.e. $\mathbf{E}_i$ must be wholly or in part

an electric field *which is not electrostatic.* The e.m.f. in a closed stationary circuit is then

$$e = \oint \mathbf{E}_t \cdot d\mathbf{l}. \tag{37b}$$

We may, however, express this as the line-integral of the *resultant* field in the conductor, $\mathbf{E}$, since $\mathbf{E}_t = \mathbf{E} - \mathbf{E}_d$ and the closed line-integral of $\mathbf{E}_d$ is zero. That is, for a stationary closed circuit,

$$e = \oint \mathbf{E} \cdot d\mathbf{l}. \tag{37c}$$

So far we have confined these definitions of e.m.f. to stationary circuits, in which the force per unit charge acting in the conductor is identical with the electric field intensity. If the conductor is moving however, this is not generally true and, in particular, if the conductor is moving through a magnetic field there arises an additional force. We therefore introduce the symbol $\mathbf{F}_t$ to denote the force acting on a unit charge considered to be inside the conductor and moving with it, but excluding the force due to the electrostatic field of the charges displaced in the conductor by $\mathbf{F}_t$. If we include the latter we obtain the resultant force per unit charge acting in the conductor, $\mathbf{F}$, and the definitions of e.m.f. in an open and a closed circuit become

open circuit
$$e_{AB} = \int_A^B \mathbf{F}_t \cdot d\mathbf{l} \tag{37d}$$

closed circuit
$$e = \oint \mathbf{F}_t \cdot d\mathbf{l} = \oint \mathbf{F} \cdot d\mathbf{l}. \tag{37e}$$

9. Convection currents.

There sometimes arise cases in which electric charges move, and so constitute an electric current, but do not owe their motion to the action of an electric field. For example, in the experiment by which H. A. Rowland proved that moving charges were attended by a magnetic field, electro-statically charged conductors on the sides of a vulcanite disc were given motion by rotating the disc. Again, the moving electrons in

a cathode-ray oscillograph, after being accelerated by an electric field between cathode and anode, pass through a hole in the anode and continue their journey to the fluorescent screen under their own momentum.

Such motion of charges, independent of an electric field, is termed a *convection* current, the essential distinction from a *conduction* current being that equation 1(31), giving the power necessary to maintain the current flow, no longer applies. In the case of the cathode-ray oscillograph, given above, the electron stream is therefore a conduction current from cathode to anode, but a convection current from anode to screen.

10. Displacement current.

In the first part of this chapter (Section 6) we discussed the concept of the "displacement" of an electric field, and mentioned that it was fundamental to the electro-magnetic theory of Maxwell.

Maxwell found that, in order to complete the symmetry of his mathematical expressions, it was necessary to postulate that an electric current always flowed in a closed circuit and was single-valued (that is, i should be a "solenoidal" quantity). Now this condition is clearly not fulfilled if our conception of current is limited to that of conduction current, for, to take a simple example, the charging current of an air condenser stops at the plates and no conduction current flows through the air between the plates. The only way in which to "close" such a circuit is to postulate that the changing field, in the dielectric (air), constitutes a current of the same instantaneous value as that of the charging conduction current in the wires.

Consider the circuit of Fig. 16, which shows a capacitor

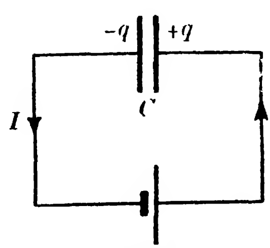


Fig 16. Charging capacitor

(to which the changing electric field is assumed to be limited: the capacitance is "concentrated" or "lumped") being charged by a battery. At any instant let the charge on the capacitor be q, and let it change by a small amount δq in time δt. Then this change of charge must be accomplished by the passage of a charge δq past any section of the connecting wires in time δt, so that the conduction current in these wires is given by

$$I_c = \frac{\delta q}{\delta t}.$$

Now since $\psi = q$ is the total displacement in the dielectric of the capacitor, it follows that we may "close" the circuit by postulating that a "displacement current", I_d, exists in the dielectric,* where

$$I_d = \frac{\delta \psi}{\delta t} = \frac{\delta q}{\delta t}, \tag{38}$$

so that I_d is equal, at every instant, to I_c.

Then since $\psi = \iint D\,dA$, where D is the displacement density at a point, and the integration is performed over a complete equipotential surface, it follows that

$$I_d = \iint \frac{\delta D}{\delta t}\,dA = \iint J_d\,dA,$$

where

$$J_d = \frac{dD}{dt}, \tag{39}$$

the displacement-current density.†

Maxwell's hypothesis of displacement current is that, whenever any electric field (electro-static or electro-dynamic)

* For a further discussion of this example, see Chapter III, Section 14 (d).

† Maxwell himself put $J_d = dD/dt$, the rationalized form, but otherwise used the unrationalized c.g.s. system. In vol. II, paragraph 610 (p. 253), of his *Treatise* he writes:

"One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current $\mathfrak{C}$, that on which the electromagnetic phenomena depend, is not the same thing as $\mathfrak{R}$, the current of conduction, but that the time-variation of $\mathfrak{D}$, the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write:

$\mathfrak{C} = \mathfrak{R} + \dot{\mathfrak{D}}$ (Equation of True Currents)."

changes, we are to suppose that a current exists in the dielectric, whose density at any point *is given by 1(39)*. This current is assumed to be attended by a magnetic field in just the same way as a conduction or convection current, and when both conduction and displacement currents are included in the total, an electric current can be said always to be single-valued and to flow in a closed circuit.

The *direction* of a displacement current is found as follows:

> If D *increases*, then the displacement current "flows" in the direction of fall of potential (or in the direction of the e.m.f.) of the field. That is, it flows in the *positive* direction of the field.

and conversely:

> If D *decreases*, the displacement current flows in the *negative* direction of the field

These statements follow at once from a consideration of the fact that, whenever current flows through a piece of electrical apparatus in the direction of fall of potential (or "voltage drop"), then energy is absorbed by that apparatus. In the case of D increasing, we have an increase of energy storage in the electric field: that is, the "apparatus" absorbs energy.

11. Polarization current.

In a solid dielectric, we had [see 1(18d)]

$$D = \epsilon_0 E + P,$$

where P is the "polarization" of the dielectric. From 1(39) the displacement current-density is

$$J_d = \frac{dD}{dt} = \epsilon_0 \frac{dE}{dt} + \frac{dP}{dt}. \tag{40}$$

The first part of this expression is due solely to the change in the net field intensity E, but the second part, dP/dt, represents a true motion of electric charges in the dielectric structure, and so in this sense might be called a conduction current. The difference between dP/dt and a conduction current, however, is that the latter can flow under a steady unchanging electric field, while dP/dt depends upon a change of field. It is called

the "polarization" current-density, since it is the rate of change of the polarization, P.

It is clear that current can flow in a path which is interrupted by an insulated portion only when the p.d. across that portion is changing. The total charge accumulated on each boundary of the insulated portion is then given by

$$q = \int I\,dt. \tag{41}$$

The displacement current affects the rate of energy conversion (that is, it is of practical importance) in a current-circuit only when the change in electric field intensity is extremely rapid. Consequently it can be entirely ignored in alternating current-circuits of power frequency, but becomes of major importance at radio frequencies, and in the classical theory of electro-magnetic radiation. The concept is, of course, only another way of stating that a changing electric field is attended by a magnetic field. A viewpoint which makes even this assumption unnecessary is discussed in Chapter v, and may be summarized as "retarded action at a distance".

12. The displacement current in a metal.

The conduction current-density in a conductor is given by $J = \sigma E$ [equation 1(35a)], where E is the electric field intensity at a point inside the conductor. Consequently, whenever the current changes, it must be due to a change in E. But a changing E constitutes a displacement current, so that we must admit that a displacement current can exist in a conductor.

The question now arises, how shall we express the "total" current-density in a metal? Following our definition of displacement current, we might be tempted to write

$$\begin{aligned} J_t &= J + \frac{dD}{dt} \\ &= \gamma E + K\epsilon_0 \frac{dE}{dt}. \end{aligned} \tag{42}$$

This expression, however, is quite inconsistent with our electro-static definition of K, whereby the value of K for a

good conductor (if K has any meaning in such a case) would approach infinity.

The difficulty, however, is fictitious. The second term in the above expression includes a polarization current-density, dP/dt [see 1(40)], which does not appear to exist in a metal. In an insulator, *polarization* current means an actual shifting of atomic charges whose *motion is limited* (i.e. a change in the *strain* of the structure) If such an effect exists in metals, it is not observable by ordinary experimental methods.

Thus, although 1(42) may be considered to be applicable to *poor* conductors (i.e. insulators with a very low, but not zero, conductivity ρ), it is better to limit the displacement current-density in a metal to the term $\epsilon_0 \dfrac{dE}{dt}$ alone. By so doing we recognize the *microscopic* viewpoint that the actual space occupied by electrons and atomic nuclei, in a solid body, is extremely small, and that in the relatively vast volume of space between them an electric field, of averaged value E, exists which has a displacement current calculated by giving to K the value for free space, that is, unity.

For a metal, then, we postulate that the "total" current-density is given by

$$J_t = \gamma E + \epsilon_0 \frac{dE}{dt} \tag{43}$$

The displacement current term, however, is of no practical importance in metals until such rapidly changing fields as those associated with visible-light frequencies are reached.* Even at the frequencies of ultra-short radio waves it may be neglected

13. Motion of isolated charges: the current element.

In applying our definition of conduction current (as the rate at which charges pass a given section of a conductor) we

* In rationalized (m.k s.) units, $\epsilon_0 = 8{\cdot}85 \times 10^{-12}$, while γ for copper is about 6×10^7. Consequently for the displacement current-density to be as high as 1 % of the conduction current-density, dE/dt must be equal to about 7×10^{16} times E. This would entail a sinusoidal alternation of E at a frequency of about 10^{16} cycles per sec. This frequency is that of a "soft" X-ray.

obtain a true *instantaneous* value of current only if we assume a continuous volume-distribution of charge throughout the conductor, for in this case alone can we reduce the time-interval of observation, in the limit, to zero. The density of charge in a metallic conductor is so enormous, compared with the charge of a single electron, that we may accept the idea of a continuous volume-distribution of charge in metals without misgiving, but if we look upon an electron as a discrete charged particle, then with extremely small currents we should expect a discontinuity in current flow under certain conditions.

Such conditions are indeed realized in practice in the case of high-gain vacuum-tube amplifiers. In a vacuum tube the current consists practically of a pure stream of electrons, and when extremely small currents are highly amplified the individual nature of the discrete electrons makes itself evident by discontinuities in the amplified current, this phenomenon usually being called the "shot effect".

For the sake of measuring the current in such a case, we can smooth out the discontinuities by taking a *mean* rate of flow of the charges. That is, we must measure the quantity of electricity passing through an area in a finite interval of time, and we take no account of the actual number of discrete charges which may be at any particular place at a definite instant. Consider the case of a column of troops marching: the *average* rate at which men pass any point of the route is a constant quantity, provided the ranks contain equal numbers of men, and are equally spaced, and provided they all march at the same speed. But the number of men at the chosen cross-section of the route *at any instant* may be equal to the number of men in one rank, or it may be zero.

Now consider a single chain of charged bodies, each having the same charge q, spaced a uniform distance δl apart and all moving along the axis of the chain with velocity v (Fig. 17).

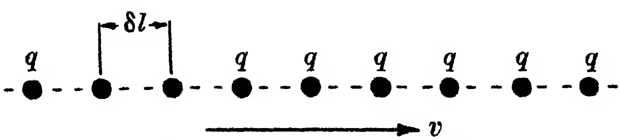


Fig. 17. Current element, $I\delta l = qv$

Then the magnitude of the charge passing any point in one second is the measure of the current, or

$$I = \frac{qv}{\delta l} = \frac{q}{\delta t},$$

where δt is the time taken for one charge q to travel the distance δl.

We may now think of the average current, I, over a length δl as being due to the motion of one only of the charges q through this distance in time δt, and the current in all other lengths δl as being due to the motion of one charge q over each length. Thus if there is only one charge, moving with velocity v, between two points δl apart, we may say that the average current flow between the two points, while the charge is moving

is $$I = \frac{qv}{\delta l},$$

that is $$I\,\delta l = qv. \tag{44}$$

Or, for a finite path L traversed by a charge q with velocity v in time T:

$$\text{Average current over path: } I = \frac{qv}{L} = \frac{q}{T}. \tag{45}$$

The current element. A current I in a very short element of path δl is called a *current element*, $I\,\delta l$. Its great value lies in finding the total effect of a finite current circuit by integrating the separate effects of all elements $I\,\delta l$ forming the circuit. If we know all the properties of a charged particle q, moving with velocity v, we can deduce the properties of a current circuit by using the relation $I\,\delta l = qv$, and then integrating around the circuit, in which process δl tends in the limit to zero.

The case of a charged particle moving in a closed orbit. As an illustration of 1(45), consider the uniform motion of a charge q in an orbit (Fig. 18). Let T be the time taken for the charge to make one complete circuit, then the amount of charge passing any section, such as P, in one second, is

$$I = \frac{q}{T},$$

which is thus the *average* current around the orbit.

The result expressed in 1(45) must not be interpreted as meaning that the effects at a given point in space, due to the moving charge, are the same at any instant as those due to a steady current I flowing in a conductor of length L. In the latter case the effects at the point are steady, while in the case of a moving charged particle they will be variable.

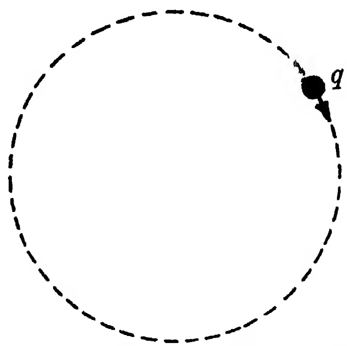


Fig. 18. Charge moving in an orbit

Equation 1(45) merely states that the total charge passing the terminals of the path in time T is equal to the average current I multiplied by the time. We shall study the instantaneous effects of a moving charged particle when we deal with the magnetic and vector-potential fields of a current.

EXAMPLES, CHAPTER I

PART I

ELECTRO-STATICS

1. If it were possible to place two isolated point-charges, each of one coulomb, at a distance of one metre apart, what would be the electro-static force between them?

Ans. 9×10^9 newtons or about 900,000 (long) tons weight.

2. If two point-charges, each of one ab-coulomb (electro-magnetic c.g.s. unit) were situated one centimetre apart, what would be the force between them in dynes? *Ans.* c^2 dynes, where $c = 3 \times 10^{10}$.

3. A charge of 10^{-6} coulomb is in a field of 5000 volts per metre (5×10^{9} ab-volts per cm.). What force does it experience?

Ans. 5×10^{-3} newtons or 5×10^{2} dynes.

4. An isolated copper sphere of radius 10 cm. is deprived of 10^{12} electrons. Find the field intensity (a) at its surface, (b) at a point one metre distant from its centre. The sphere is situated in air ($K=1$).

Solution. In 1(10) we put $q=1{\cdot}602 \times 10^{-7}$ coulomb, and since

$$\epsilon_0=8{\cdot}854 \times 10^{-12} \text{ we get}$$

(a) $r=0{\cdot}1$ metre, whence $E=1{\cdot}44 \times 10^{5}$ volts per metre,

(b) $r=1{\cdot}0$ metre, whence $E=1{\cdot}44 \times 10^{3}$ volts per metre

5. A parallel-plate condenser has capacitance C, and is charged to a p d V. If d is the distance between the plates, prove that the force of attraction between them is given by

$$F=\frac{CV^2}{2d}.$$

6 A parallel-plate condenser has plates 20 cm. square and 0·5 cm. apart. Find

(a) its capacitance,

and, when charged to a p.d. of 1000 volts:

(b) its charge,

(c) the energy stored,

(d) the force of attraction between the plates

Solution ($K=1$ for air) (a) From 1(23),

$$C=\frac{\epsilon_0 A}{d} \qquad \begin{pmatrix} A=0{\cdot}04 \text{ sq metre,} \\ d=0{\cdot}005 \text{ metre} \end{pmatrix}$$

$$=\frac{8{\cdot}854 \times 10^{-12} \times 0{\cdot}04}{0{\cdot}005}$$

$$=70{\cdot}8 \times 10^{-12} \text{ farad}$$

$$=70{\cdot}8 \text{ picofarads (pF)}$$

(b) $Q=CV=70{\cdot}8 \times 10^{-12} \times 10^{3}=7{\cdot}08 \times 10^{-8}$ coulomb.

(c) $W=\frac{1}{2}CV^2=3{\cdot}54 \times 10^{-5}$ joule.

(d) $F=\dfrac{CV^2}{2d}=\dfrac{70{\cdot}8 \times 10^{-12} \times 10^{6}}{0{\cdot}01}=7{\cdot}08 \times 10^{-3}$ newton (708 dynes).

7. State the law governing the electric flux (i.e. total displacement) over a surface surrounding a charge of electricity, and apply it to deduce an expression for the capacity of a plate capacitor.

(a) Find the capacity of a capacitor consisting of two parallel metal plates each having an area of 1500 sq. cm. and separated by a layer of air 0·5 cm. thick. Neglect the capacity between the external surface of the plates.

(b) When a sheet of ebonite 0·18 cm. thick is introduced between the plates and their distance apart increased to 0·61 cm. it is found that

the capacity remains unaltered. Determine the capacity of the condenser with the ebonite in position and a spacing of 0·5 cm. between the plates. (London, External B.Sc., 1935.)

Solution. (a) From 1(23),

$$C = \frac{\epsilon_0 A}{d} = \frac{8{\cdot}854 \times 10^{-12} \times 0{\cdot}15}{5 \times 10^{-3}}$$

$$= 2{\cdot}656 \times 10^{-10} \text{ farad}$$

$$= 265{\cdot}6 \text{ pF}.$$

(b) First we shall find the capacitance of a parallel-plate capacitor, whose plates are distant d apart, and between which there is a sheet of insulating material of thickness d_1 and dielectric constant K.

The arrangement is equivalent to an air-capacitor C_1, of spacing $(d-d_1)$, in series with a capacitor C_2, of spacing d_1, whose dielectric constant is K, the two capacitors having the same area.

From 1(28), if C is the total capacitance,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{(d-d_1)}{\epsilon_0 A} + \frac{d_1}{K\epsilon_0 A}$$

$$= \frac{1}{\epsilon_0 A}\left\{d - d_1 + \frac{d_1}{K}\right\}.$$

The capacitance without the insulating sheet is

$$C_a = \frac{\epsilon_0 A}{d},$$

so that the ratio
$$\frac{C_a}{C} = \frac{Kd-(K-1)d_1}{Kd}. \qquad (46)$$

In this problem we shall first find K for the sheet of ebonite. When the sheet is introduced and the spacing of the plates increased, the charge is unaltered, and since C is unaltered so is the p.d. between the plates. The field intensity E in the air space is unaltered since the charge is constant.

Let d be the original spacing of the plates $= 0{\cdot}5$ cm.,
d_1 be the thickness of the ebonite sheet $= 0{\cdot}18$ cm.,
d_2 be the thickness of the air space to keep the capacitance unchanged $= 0{\cdot}43$ cm.

Then the p.d., V, $= dE = d_2 E + \dfrac{d_1 E}{K}$,

whence
$$K = \frac{d_1}{d-d_2}$$

$$= 2{\cdot}57.$$

The ratio
$$\frac{C_a}{C} = \frac{2{\cdot}57 \times 0{\cdot}5 - 1{\cdot}57 \times 0{\cdot}18}{2{\cdot}57 \times 0{\cdot}5} = \frac{1}{1{\cdot}283},$$

so that
$$C = 1{\cdot}283 \times C_a = 1{\cdot}283 \times 265{\cdot}6 = 340{\cdot}8 \text{ pF}.$$

8. The potential difference between an isolated sphere 5 cm. in diameter and earth is 10,000 volts.

Find the charge on the sphere, the electric field (intensity) near the surface of the sphere and the electric field and the electrical potential at a distance of 20 cm. from the centre of the sphere.

(London, External B.Sc., 1933.)

Ans. $2{\cdot}782 \times 10^{-8}$ coulomb, 400,000 volts per metre, 6250 volts per metre, 1250 volts to earth.

9. An air-capacitor consists of two concentric cylinders, fitted with guard cylinders. The inner and outer diameters of the outer and inner cylinders respectively are 0·493 and 0·295 metre, the axial length being 0·762 metre. Calculate the capacitance of the condenser.

Ans. 82·7 pF

10. A slab of insulating material, 4·2 mm. thick, is introduced between the plates of a parallel-plate capacitor. To restore the capacity of the capacitor to its original value, it is found to be necessary further to separate the plates by 2·3 mm. Find the dielectric constant of the material of which the slab is made. (See Ex. 7 above.)

(London, External B.Sc., 1934.)

Ans. 2·21

11. A metal sphere of radius r is situated concentrically inside a hollow metal sphere of inner radius R. If the spheres are charged to a p.d. V, show that the potential gradient in the dielectric between the spheres is a maximum at the surface of the inner sphere, and is given by

$$\mathscr{E}_{\text{max}} = \frac{VR}{r(R-r)}.$$

If R is fixed, and r variable, show that, for a constant p.d., this maximum potential gradient is a minimum when $r = R/2$.

Part II

The Electric Current

12. What is the velocity of drift of the moving electrons in a metal conductor, if the current density is 100 amps per sq. cm. (10^6 amps per sq. metre), and the moving charge, per unit volume, is 5000 coulombs per cubic cm. (5×10^9 coulombs per cubic metre)?

Ans. 2×10^{-4} metre per sec. (2×10^{-2} cm. per sec.).

13. A hot-water tank contains 20 gallons of water whose temperature is to be raised from 20° C. to 60° C. by an electric heater. Assuming no heat loss, find the electrical energy required, in joules.

If the resistance of the heater is 10 ohms, and the p.d. applied to it 100 volts, how long must the current flow, assuming the resistance of the heater to remain constant? *Ans.* $1{\cdot}52 \times 10^7$ joules, 4·23 hours.

14. If ρ for copper is $1{\cdot}6 \times 10^{-8}$ ohm-metre ($1{\cdot}6 \times 10^{-6}$ ohm-cm.), what is the resistance of a copper conductor of area 10^{-4} sq. metre (1 sq. cm.) and 1000 metres (10^5 cm.) long?

(Verify that equation 1(34) is true for both metre and centimetre units.) *Ans.* 0·16 ohm.

15. If the current-density in the conductor of Ex. 14 is 2×10^6 amps per sq. metre (200 amps per sq. cm.), what is the electric field intensity (e.m.f. gradient) along the conductor?

(Verify that 1(35) is true for both metre and centimetre units.)

Ans. $3{\cdot}2 \times 10^{-2}$ volt per metre ($3{\cdot}2 \times 10^{-4}$ volt per cm.).

16. If the conductivity, γ, of copper is $6{\cdot}2 \times 10^7$ mhos per metre ($6{\cdot}2 \times 10^5$ mhos per cm.), and the field intensity in a conductor is 2×10^{-2} volt per metre (2×10^{-4} volt per cm.), what is the current-density?

(Verify that 1(35a) is true for both metre and centimetre units.)

Ans. $1{\cdot}24 \times 10^6$ amps per sq. metre ($1{\cdot}24 \times 10^2$ amps per sq. cm.).

17. Taking the conductivity of copper as in Ex. 16, find the conductance of a conductor 1000 metres (10^5 cm.) long, of sectional area 10^{-4} sq. metre (1 sq. cm.).

Verify that 1(35b) is true for both metre and centimetre units.

Ans. 6·2 mhos.

18. Find an expression connecting the temperature coefficients of a conductor when reckoned from two different temperatures.

If the temperature coefficient of copper at 0° C. is 0·00428, what is the resistance at 50° C. of a copper coil whose resistance at 20° C. is 250 ohms? (London, External B.Sc., 1933.)

Ans. 280 ohms.

19. The temperature of a given copper conductor is the same as that of the surrounding air and is equal to T_c degrees Centigrade. The resistance of the conductor at this temperature is R_c ohms.

After the passage of an electric current for some time the resistance of the conductor is found to be R_h ohms, while the temperature of the surrounding air is now T_a. If the temperature coefficient of copper at 0° C. is equal to 1/234·5, prove that the rise in temperature of the conductor, above the ambient temperature, is equal to

$$\frac{R_h}{R_c}(234{\cdot}5 + T_c) - (234{\cdot}5 + T_a) \text{ degrees C.}$$

The resistance of a copper conductor at zero degrees C. is 3·25 ohms. What is its resistance at 100° C. if the temperature coefficient of copper at 50° C. is 0·0035? *Ans.* 4·63 ohms.

20. The resistance of a transformer winding at 15° C. was 0·0155 ohm. After the transformer had been loaded for some time the resistance of the winding was found to be 0·0178 ohm, and the air temperature had risen to 18° C. Find the rise in temperature of the winding due to the load current. (See Ex. 19.) *Ans.* 34° C.

21. A uniform electric field (in air), of intensity one million volts per metre, collapses at a uniform rate: (a) in one milli-second, and (b) in one micro-second. What is the displacement current-density in each case?

Ans. (a) 8·854 milli-amps per sq. metre,
(b) 8·854 amps per sq. metre.

22. By considering a cubic metre of a dielectric in a uniform field (two opposite faces of the cube being equipotential surfaces), deduce equation 1(21), for the energy stored in the field, by integrating the instantaneous power expended (power = product of the p.d between equi-potential surfaces and the displacement current through the cube), as the field intensity rises from zero to its final value.

23 An electron, starting from rest, moves unimpeded in an electric field of intensity E volts per metre Find

(a) the force it experiences,
(b) its acceleration,
(c) the kinetic energy it attains in moving through a potential difference of V volts,
(d) the velocity it attains in moving through a p d of V volts.

(Take $q = 1{\cdot}602 \times 10^{-19}$ coulomb, mass $= 9{\cdot}11 \times 10^{-31}$ kilogram.)

Solution (a) The force acting on the electron, from 1(3), is

$$F = Eq = (1{\cdot}602 \times 10^{-19})\,E \text{ newtons}$$
$$= (1{\cdot}602 \times 10^{-14})\,E \text{ dynes}$$

(b) The acceleration, $\alpha = \dfrac{\text{force}}{\text{mass}}$

$$= \frac{1{\cdot}602 \times 10^{-19}}{9{\cdot}21 \times 10^{-31}}\,E = (1{\cdot}758 \times 10^{11})\,E \text{ metres per sec}^2,$$

or, in c g.s. units,

$$= \frac{1{\cdot}602 \times 10^{-14}}{9{\cdot}12 \times 10^{-28}}\,E = (1{\cdot}758 \times 10^{13})\,E \text{ cm per sec.}^2.$$

(c) By the definition of p d., the work done when an electron moves through V volts is Vq joules. This is converted into the kinetic energy of the electron. Neglecting the relativity change of mass at high velocities we get

$$\tfrac{1}{2}mv^2 = (1{\cdot}602 \times 10^{-19})\,V \text{ joules}$$

or

$$= (1{\cdot}602 \times 10^{-12})\,V \text{ ergs}.$$

(d) From (c), $\frac{1}{2}mv^2 = qV$, so that

$$v = \sqrt{\frac{2qV}{m}}$$

$$= \sqrt{\frac{2 \times 1{\cdot}602 \times 10^{-19}V}{9{\cdot}12 \times 10^{-31}}} = 5{\cdot}93 \times 10^5\sqrt{V} \text{ metres per sec.}$$

or, in c.g.s. units,

$$v = \sqrt{\frac{2 \times 1{\cdot}602 \times 10^{-12}V}{9{\cdot}12 \times 10^{-28}}} = 5{\cdot}93 \times 10^7\sqrt{V} \text{ cm. per sec.}$$

Note. The mass m of the electron can be assumed constant only if $v^2 \ll c^2$. If we retain the usual equations of Newtonian dynamics, the theory of relativity shows that the mass that must be used in the equation

$$\text{mass} = \frac{\text{force}}{\text{acceleration}},$$

when the force is in the direction of the velocity v, has the value

$$m_L = \frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

(where $m = 9 \cdot 11 \times 10^{-31}$ kg.), and is called the "longitudinal mass".

If the force is transverse to the velocity v (as in the case of a cathode-ray oscillograph whose beam is deflected by transverse forces) the equivalent mass has the value

$$m_t = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

and is called the "transverse mass".

(See, for example, H. A. Lorentz, *Lectures on Theoretical Physics*, III, p. 230.)

The results worked out above therefore apply only to comparatively low values of the voltage V.

24. A stream of electrons is moving at a constant speed of 10^6 cm. per sec. between two electrodes in a vacuum chamber. If the number of electrons per centimetre length of the stream is 10^{12}, what current is flowing between the electrodes? *Ans.* 0·1602 amp.

25. If, in Ex. 24, it is assumed that there are also 10^{12} positive ions, each having unit electronic charge, per centimetre length of the stream, moving at a constant speed of 500 cm. per sec., in a direction opposite to that of the electrons, what is the current in this case?

Ans. 0·1603 amp.

26. A single electron moves in a circular orbit of radius $2 \cdot 11 \times 10^{-8}$ cm. at a speed of $1 \cdot 09 \times 10^8$ cm. per sec. What is the average current around the orbit? *Ans.* 0·132 milli-amp.

CHAPTER II

THE MAGNETIC FIELD AND ELECTROMAGNETIC INDUCTION

PART I

INTRODUCTION

The origin of the word "magnet" may be traced to Magnesia, where according to Lucretius (95–52 B.C.) a peculiar substance was found which exerted mysterious attractive forces upon pieces of iron. Now long before the discovery of the dependence of "magnetism" upon electric currents, these forces had been put to practical use in the mariner's compass, and by the end of the first quarter of the nineteenth century a comprehensive theory of magnetism, due chiefly to Coulomb and Poisson, had been built up, based on the inverse-square law which we have seen to be the basis of electro-statics.

The force on a compass needle appears to act on regions near the ends which are termed the "poles", and the inverse-square law of magnetic attraction and repulsion is based, as in electro-statics, on the hypothesis that the phenomena originate at dimensionless points. Such a hypothetical "magnetic pole" is analogous to a "point-charge", and again the mutual forces are supposed to be transmitted through the intervening space by a peculiar property of that space which we call the "magnetic field".

We cannot consider here the various theories that were put forward, in early days, to account for electrical and magnetic phenomena. In the eighteenth century the nature of the structure of matter was quite unknown, and although certain facts, such as the magnetization of a needle by a lightning stroke, suggested to enquiring minds some connection between electricity and magnetism, the phenomena were looked upon as being distinct. Magnets were found in nature, so it was not unnatural to think of magnetism as being of a fundamental

nature and at the time of the formulation of the theories of Coulomb and Poisson there was no reason to suppose that a "unit magnetic pole" was any less real or fundamental than a "unit electric charge".

The discoveries of the last century, however, have shown conclusively that magnetic phenomena are due to electric currents. Oersted in 1820 discovered that an electric current sets up a magnetic field, for he observed that a small compass needle was deflected in the vicinity of a wire carrying a current, and this discovery was later put into perfect quantitative form by Ampère. An immediate consequence was the realization that a wire carrying current and situated in a magnetic field should experience a force, a phenomenon experimentally observed by Faraday in 1821. There followed an intensive search for the complementary phenomenon: the production of electric currents from magnetism, a search which for several years proved fruitless since experimenters expected to find electrical effects in a stationary arrangement of magnet and conducting circuit. The truth was finally disclosed when Faraday in 1831 found that, for the production of an electric current by magnetism, it is necessary that some relative change (either of magnitude or position) should take place between the circuit and the magnetic field.*

Clerk Maxwell (1831–79) used Faraday's experimental work as the basis of his classical theory of electro-magnetism, from which he deduced the electro-magnetic nature of light and predicted the propagation of electro-magnetic waves by high-frequency alternating currents. In so doing he took Faraday's discovery of electro-magnetic induction and expressed it by stating that an electric field (i.e. an e.m.f.) exists in the space where a magnetic field is changing, and then made the bold assumption that the complement should be true; namely that a magnetic field should exist in the space where an electric

* Every student of electrical engineering should read the papers of J. J. Fahie and W. Cramp, delivered at the Faraday Centenary Celebrations in 1931, and published in the *Jl. I.E.E.* LXIX (1931), pp. 1329 and 1357, under the titles: "Magnetism, electricity, and electro-magnetism up to the time of the crowning work of Michael Faraday in 1831", and "The Birth of Electrical Engineering".

field is changing. Expressed in mathematical form, these relations combined to produce the equations of wave-motion which have been so effectively verified in the development of radio communication, thereby proving the truth of this initial assumption, that the magnetic effects of both conduction and displacement currents are the same.

Given the true laws of dependence of magnetic phenomena upon electric currents, it follows that any laws governing the configuration of magnetic fields should be obtainable from the laws already known. This was not the sequence of discovery, however, for the inverse-square law of magneto-statics, and the concepts of point-charges and point-poles, arose long before Maxwell's co-ordinating theory, in which he made valuable use of these concepts.* This is perhaps the reason why the magnetic pole has such a tenacious hold on life, and why the usual presentation of the subject still uses magnetism as the starting-point, in spite of the fact that in the modern theory of matter magnetism is not a fundamental phenomenon *per se*, but purely an aspect of electricity in motion.

Following his quantitative formulation of Oersted's discovery, Ampère put forward the bold suggestion that the peculiar phenomena of the magnetization of iron might be due to electric currents in its structure ("molecular currents"). It should be remembered that this was long before the discovery of the electron, and the electronic theory of matter, and was made in order to provide a single and comprehensive origin for all magnetic fields. The discovery of the electron by J. J. Thomson, the formulation of electron theory by H. A. Lorentz, and the increasing knowledge of atomic structure which has been the great scientific romance of the present

* But he also wrote: "According to Ampère's hypothesis...the properties of what we call magnetized matter are due to molecular electric circuits, so that it is only when we regard the substance in large masses that our theory of magnetization is applicable, and if our mathematical methods are supposed capable of taking account of what goes on within the individual molecules, they will discover nothing but electric circuits, and we shall find the magnetic force and the magnetic induction everywhere identical" (*Electricity and Magnetism*, vol. II, Article 615).

century, have all combined to add experimental justification to Ampère's hypothesis. Although there are many details of magnetic phenomena which are yet imperfectly understood, it would seem that the main aspects of the electrical foundation of magnetism are well established. Atoms, for instance, are known to have magnetic moments due to orbital and spinning electrons,* and the magnetic field of iron is considered to be due to the latter. The names of Weber, Ewing, Langevin, Weiss and others stand out as pioneers in the endeavour to reconcile the complicated magnetic properties of matter in bulk with the fundamental concepts of electricity and electron theory.

In the following pages an exposition of the fundamental relations between electricity and magnetism is built upon the discoveries of Oersted and Faraday.

PART II

PRELIMINARY DISCUSSION OF SOME ASPECTS OF OERSTED'S DISCOVERY

1. Magnets and the magnetic field.

The force experienced by a small compass needle in the vicinity of a permanent magnet varies in magnitude and direction from point to point. As in the case of the proximity of charged bodies, we postulate a special state of the intervening space whereby the mutual forces are transmitted, and call this state the "magnetic field". We again map out the field by means of imaginary "lines of force", the direction of which at any point being tangential to the length of a small compass needle placed at the point, and the density of which gives a measure of the "strength" of the field, or of the forces which the ends of the compass needle experience in the field.

Now a compass needle earns its name from its property of aligning itself with the magnetic field of the earth, and that

end which points towards the north is called the "north-seeking" (or simply the "north") pole, the other end being the "south-seeking" or "south" pole. If the field of a bar-magnet is mapped out by moving a compass needle into various positions, it is found that the "lines of force" always begin and end on the magnet, in the vicinity of its ends, and the *positive* direction of a line of force is taken as the direction of the axis of the needle from its south to its north pole, or the direction from the north to the south pole of the bar-magnet. The well-known rule that like magnetic poles repel, and unlike

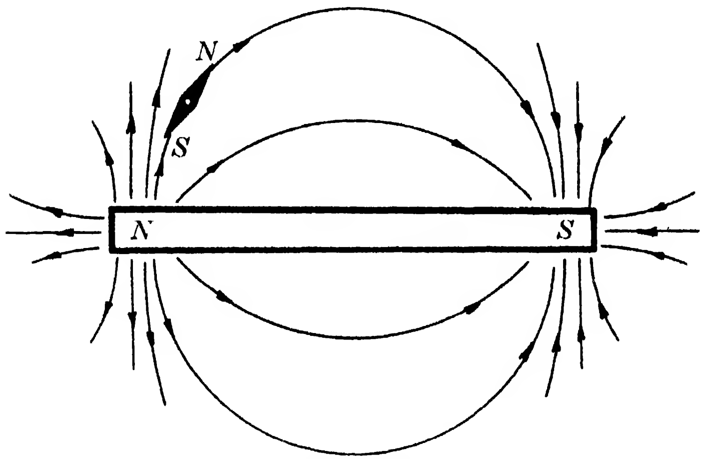


Fig. 19. Field of bar-magnet

poles attract, one another is similar to that governing the mutual forces of electric charges, and gave rise to the early conception of magnetism as being concentrated at points (or poles) within the magnet, near its ends, from which the lines of force were supposed to emanate.

2. The magnetic field due to a current in a straight conductor.

Oersted discovered that when a compass needle is situated with its rotational axis perpendicular to a wire carrying an electric current, there is a tendency for it to set itself at right angles to the axis of the wire, and that a definite relation exists between the relative positions of the north and south poles of

the needle, and the direction of the current. Thus in Fig. 20, the circle with a cross represents the cross-section of a conductor, perpendicular to the plane of the paper, with a current flowing downwards. A compass needle, with its axis AB perpendicular to the wire and in the plane of the wire's axis, will set itself (if shielded from all other magnetic fields) in the plane of the paper, with its north pole to the right and its south pole to the left, and we find that the torque on the needle is proportional to the current in the wire. If, however, a compass needle is placed with its axis at CD, in the plane of the paper, in such a way that the needle can rotate only in a plane passing

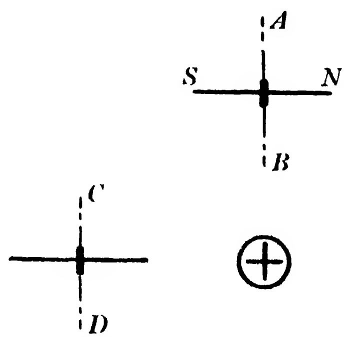


Fig. 20. Compass needles near a straight current

through the axis of the conductor, it will experience no torque about its axis.

We interpret these results by saying that a straight conductor carrying current, and remote from the return path of the current, sets up a magnetic field whose lines of force are circles concentric with the axis of the conductor and in planes perpendicular to this axis, and that the magnitude of this field at any point is proportional to the current in the conductor.

There is no component of magnetic field parallel to the axis of the wire, and the positive direction of the lines of force is clockwise when viewed in the direction of current flow. This gives the well-known *corkscrew rule*:

> The direction of the magnetic field of a current in a straight conductor bears the same relation to the direction of current flow as the direction of progression along the

thread of a right-handed screw bears to the direction of progression along its axis.

Fig. 21 shows this relation graphically, the current flowing downwards in (*a*) and upwards in (*b*).

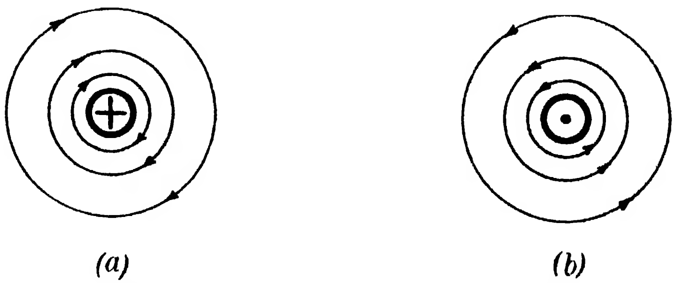


Fig. 21. Magnetic field of a straight current

3. The magnetic field of a circular current.

Consider a circular conductor (Fig. 22) carrying a current flowing in a clockwise direction when viewed from below. (We may express this more simply in the language of dynamics by saying that the rotational axis of the current is upwards.)

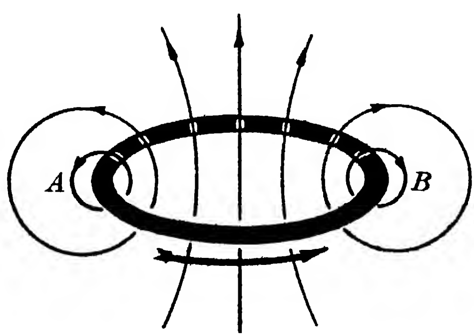


Fig. 22. Magnetic field of a circular current

Close to the surface of the wire the lines of force will be approximately circular, as at A and B, and the direction of the field *inside* the loop is upwards. Nearer the axis of the ring

we cannot say that the lines of force are circles, but we can deduce from the corkscrew rule that the field inside the loop must be everywhere *upwards*, or in the direction of the rotational axis of the current. The whole arrangement is symmetrical about the axis of the coil, so there is no reason for the field along this axis to curve one way rather than another, hence we deduce the fact that the field at the axis is linear and coincident with the axis. We may then call the axis of the coil the *magnetic axis*.

The relation between the direction of a circular current and the field it produces along the axis of the coil is thus exactly the same as that between the field set up by a current in a straight conductor and the direction of that current. We also note that the field of the coil is similar in configuration to that of a magnetized plate or shell, having north polarity on its upper surface and south polarity on its lower surface.

4. Mutual force between two parallel straight conductors.

Now since magnets experience mutual forces, and since electric currents are equivalent to magnets in that they set up magnetic fields, it follows that neighbouring conductors carrying currents should experience mutual forces. It is found by experiment that two parallel straight conductors carrying currents I_a and I_b in the same direction (Fig. 23) attract one another with a force that is proportional to the product of the two currents, while if the currents flow in opposite directions the force is one of repulsion.

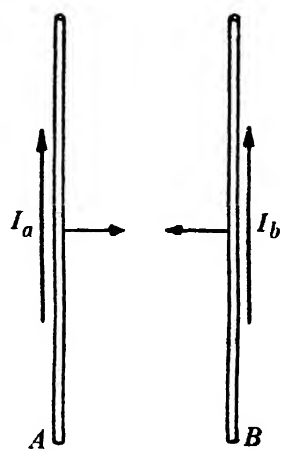


Fig. 23. Force between parallel conductors

This simple fact is really the fundamental basis of magnetism. It shows that moving electric charges experience mutual forces in virtue of their motion,* and we may look on the

* See, however, Chapter III, Section 7.

magnetic field as a hypothesis which enables us to simplify the calculation of the forces between moving charges, just as the electro-static field is a hypothesis whereby we simplify the calculation of the forces between charges at rest. We may summarize the line of thought in the two parallel cases thus:

	Electric field	Magnetic field
Observed phenomenon:		
	Force between charges A and B at rest	Force between conductors A and B
The calculation is simplified by the hypothesis that:		
	(*a*) Charge A is attended by an electric field	(*a*) Charges moving in A are attended by a magnetic field
	(*b*) Charge B, situated in this field, experiences a force, and vice versa	(*b*) Charges in B, moving in this field, experience a force, and vice versa*

(In both cases the field at any point is equal to the vector sum of the fields due to A and B separately, but since B does not experience any force due to its own field, we use the force on B as a measure of the field due to A.)

We accept, then, the magnetic field as a working hypothesis in terms of which we shall study the mutual effects of moving charges, and we also accept the definition of direction of a magnetic field based on the behaviour of magnetic needles. By so doing we are led into a whole series of relations between mutually perpendicular vector quantities.

5. Force on a conductor carrying current in a magnetic field.

Consider the directive relations between the current in conductor B (Fig. 24), the force it experiences, and the magnetic field, due to A, at the axis of B. By the corkscrew rule, the field due to the current in A at the axis of B is perpendicular to the paper and downwards. We denote the strength or density of this field by the symbol **B** (yet to be

* The force and the magnetic field are measured by the observer relative to whom the charges are moving.

defined) and the relation between the directions of B (field), I (current), and F (force) are shown in perspective in the figure, where they are seen to be mutually perpendicular.

Now it is important that the engineer should have a simple working rule whereby he can obtain the direction of the force on a wire, when carrying current in a magnetic field, and we give this rule in two forms:

A. *The vectors I, B, F form a right-handed set of mutually perpendicular vectors.* This means that if we take the first vector of the set (I) and rotate it towards the second (B), then

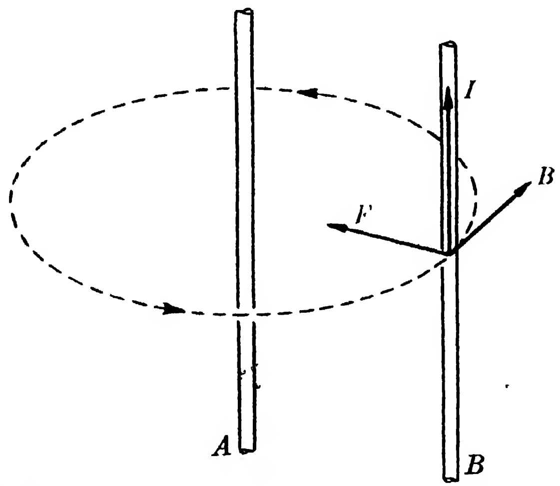


Fig. 24. Force on a conductor situated in a magnetic field

if we imagine that we are rotating a right-handed screw the direction of the third vector (F) is given by the direction of motion of the screw along its axis.

B. *Fleming's left-hand rule for motor action.* Another useful method of finding the direction of the force on the conductor is the left-hand rule of Fleming.

Hold the left hand with the thumb, first and second fingers mutually perpendicular (Fig. 25). Then they represent the directions of the vectors I, B, and F as shown.

Instead of the letters I, B, and F, it is helpful to use E, F, and M, which are in alphabetical order and may be used again

with similar significance in the right-hand rule for generator action. These letters denote:

E: electricity (current or e.m.f.),
F: field (magnetic),
M: motion (under action of force),

and E, F, and M form a right-handed set.

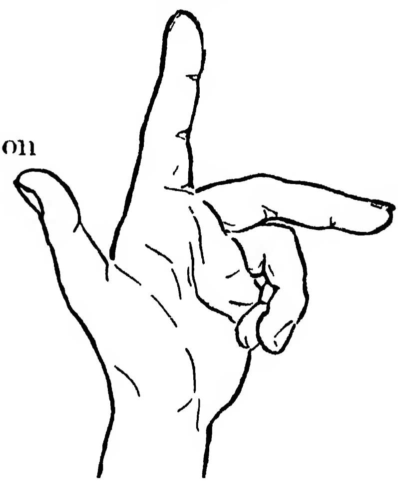


Fig. 25. Fleming's left-hand rule for motor action

PART III

PRELIMINARY DISCUSSION OF FARADAY'S DISCOVERY

1. Electro-magnetic induction.

As soon as Oersted discovered that an electric current produces magnetism, experimenters in many parts of the world began to seek the complementary relation: the production of electric currents from magnetism. It fell to the outstanding genius of Faraday to discover, in 1831, the true conditions under which such "electro-magnetic" induction of currents can take place. He found that for a current (and consequently an e.m.f.) to be induced in a circuit, it is necessary for the

circuit *either* to surround a magnetic field which is changing in strength, *or* to move in a suitable manner in the space occupied by a magnetic field.

Faraday made great use of the conception of lines or tubes of force in a magnetic field, along which he imagined a definite physical state which we now call the "magnetic flux", analogous to the displacement (ψ) in the electric field, and the quantitative properties of the field have been developed in terms of this flux. It is a measure of the total "magnetism" from a given coil or magnet, and must be considered (as will be shown later) to exist in a closed path, which in the case of a coil carrying current links the coil, and in the case of a permanent magnet passes through the iron. The magnetic field can be mapped out, as in the case of the electric field, by means of lines of force and equipotential surfaces, and the quantity of flux passing through unit area of an equipotential surface is taken as a measure of the "density" of the magnetic field, or the "induction".*

Now the density of a magnetic field controls the forces experienced by the "poles" of a small compass needle, and in the classical theory of magnetism magnetic fields are measured in terms of these forces, starting from a definition of "unit magnetic pole" based on the inverse-square law of force, and analogous to the unit point-charge. In the light of present-day knowledge, however, the concept of a unit pole seems artificial; it has no physical existence and if used should be considered as purely a mathematical concept. Magnetism, as has been

* We speak here of the "density" of the magnetic field, for which we use the symbol B. This is the real "strength" of the field, and when we speak of "lines of force" we are adhering to Faraday's original use of the term. For the sake of mathematical symmetry, however, classical theory invents a "cause" of the flux, called the magnetic "force" or "intensity", H, and it is this that is supposed (in conventional theory) to be represented by "lines of force". The "effect" of the magnetic force is then taken as B, the flux density, which is represented by "lines of induction". The viewpoint taken in this book is that H is merely a certain measure of the flux-density, called the "m.m.f. gradient". The true "cause" of B is taken to be electric charges in motion, and the function of H is to aid in calculation. In m.k.s. units H is *not* numerically equal to B in free space.

repeatedly stated, is a consequence (or aspect) of electric currents, and it is far more satisfying to a practical mind to base the magnetic definitions upon fundamental electrical operations.

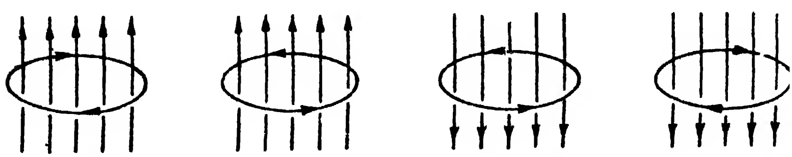
Faraday's discovery enables us to do this. In the case of a rigid coil it is found that *the e.m.f. induced by magnetic action is always proportional to the rate at which the magnetic field, linking the coil, changes*. We therefore define magnetic flux in the following way:

DEFINITION OF FLUX — Magnetic flux is a distributed vector quantity, such that when the magnitude of this quantity linking a closed path changes, an e.m.f. (i.e. an electric field) appears around the path.

2. Direction of e.m.f.: Lenz's Law.

If the closed path is a conducting wire, the induced e.m.f. will cause a current in the wire, and this current will possess a magnetic field.

Now this magnetic field, due to the induced current, is always in such a direction as to *oppose the change of the magnetic field which induces the e.m.f.* This relation, which is really a consequence of the conservation of energy, is of fundamental importance and is known as *Lenz's Law*. To illustrate the law, consider Fig. 26. A horizontal loop of wire is placed in a



ϕ increasing (*a*) ϕ decreasing (*b*) ϕ increasing (*c*) ϕ decreasing (*d*)

Fig. 26. Illustrating Lenz's law

vertical magnetic field which is changing in magnitude. In the four cases shown, the flux is directed *upwards* in (*a*) and (*b*), and *downwards* in (*c*) and (*d*). In two of the cases the flux is increasing, and in the others is decreasing. The direction of

the induced e.m.f. is given by the arrows on the loop in each case. We may tabulate the reasoning thus:

Case	Induced current must produce a field inside coil	Direction of induced current (and e.m.f.) viewed from below the coil
(*a*)	Downwards	Counter-clockwise
(*b*)	Upwards	Clockwise
(*c*)	Upwards	Clockwise
(*d*)	Downwards	Counter-clockwise

3. Methods of inducing an e.m.f.

There are two distinct methods whereby an e.m.f. may be induced in a *rigid coil*:

(A) The coil must surround the whole or part of a magnetic flux which changes in magnitude.

(B) There must be relative motion between the coil and the source of a constant magnetic field, in such a way that the amount of magnetic flux passing through the coil changes.

In both cases the e.m.f. is found to be proportional to the rate of change of the linking flux. As practical examples we may take the transformer and the rotating-armature dynamo-electric machine.

Method (*A*). *The Transformer*

Two coils are wound around a closed iron core* (Fig. 27) and

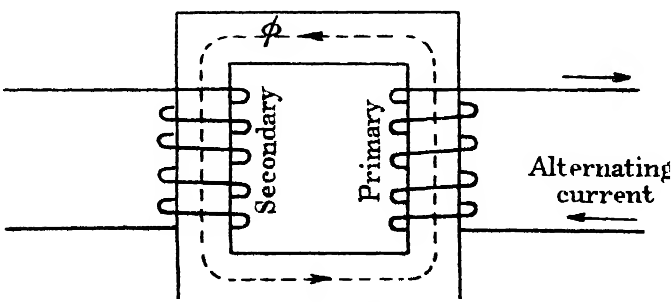


Fig. 27. Circuits of a transformer

* The presence of the iron core is not essential to the phenomenon

one of these coils (the "primary") carries an alternating current. In this case practically the whole of the magnetic field set up by this current is *inside* the iron. The wires of the coils themselves are situated in a very weak field, which for the purpose of this example may be neglected.

The alternating current in the primary causes an alternating flux ϕ in the iron core which links with the second ("secondary") coil and, in changing, induces an alternating e.m.f. in this coil. Notice that there is no relative motion between the two coils, and it is not necessary for the wires of the secondary coil to be situated *in* the magnetic field of the primary.

Method (B) The Dynamo-Electric Machine

A rigid coil is arranged to rotate between the poles of an electro-magnet, the field of which is produced by coils wound around the poles and carrying a steady direct current. The flux passes from the north to the south pole, and the coil rotates *in* this field (Fig 28)

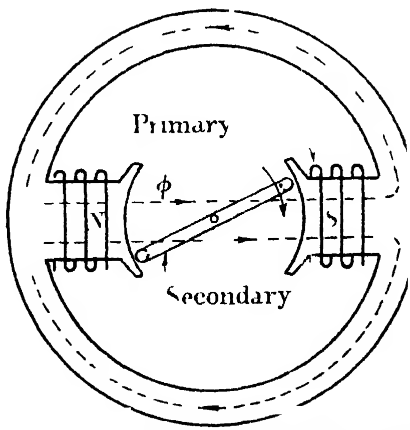


Fig. 28 Circuits of a dynamo-electric machine

An alternating e.m.f. is induced in the rotating coil, the direction of which, at any instant, can be determined by means of Lenz's Law. The magnetic field may, of course, be due to a system of permanent magnets, and in either case it is due to a system of currents which we may call the primary, relative to which the rotating coil (or secondary) moves.

Discussion of the examples

Although the results in these two cases are similar, there is a fundamental difference in *method.* In case (A) there is no relative motion between primary and secondary circuits, and the e.m.f. is due to *a changing current in the primary.* The e.m.f. in case (B), however, is due to *relative motion between primary and secondary,* the primary current remaining constant.

We may now recall two crucial experiments of Faraday, in 1831. In the first* he wound two coils on to an iron ring, one coil, A, being connected to a battery, and the other, B, to a long loop of wire passing close to a compass needle. On starting or stopping the current in coil A he noticed a deflection of the compass needle, this being evidence that an electric current had passed through the coil B. The "mechanism" of this experiment is that of the transformer.

Later,† he constructed a helical coil whose circuit he completed by a wire near which he again placed the compass needle. Into this hollow coil he plunged a cylindrical bar magnet, and noticed a deflection of the needle. Whenever the magnet was moved, a deflection occurred, but with the magnet at rest in any position within the coil there was no deflection. The mechanism here is that of our case (B), where there is relative motion between the primary (the magnet) and the secondary (the helix).

From the results of this experiment Faraday postulated that, in the case of relative motion of coil and magnet, the wires of the coil must cut across the lines of force of the field, from which reasoning he succeeded in producing an e.m.f. by rotating a disc between the poles of a large magnet,‡ a case which we shall now examine.

(C) *The case where there is no change of flux through the circuit, which always retains the same shape, but in which there is relative motion between its parts.*

* *Faraday's Diary* (Bell), Aug. 29, 1831.
† *Ibid.* Oct. 17, 1831.
‡ *Ibid.* Oct. 28, 1831.

Faraday arranged a copper disc to rotate in the magnetic field of a large electro-magnet (Fig. 29). If a circuit is completed, as shown, by connecting a wire from the bearing of the disc to a sliding contact on the periphery, a current will flow in this circuit when the disc is rotated.

Now although we can measure the e.m.f. in the *whole of the circuit* only, it seems reasonable to suppose that the stationary parts have no e.m.f. induced in them, and that the cause of the e.m.f. is the rotation of the disc in the magnetic field. The circuit as a whole may be linked by a magnetic field, but this

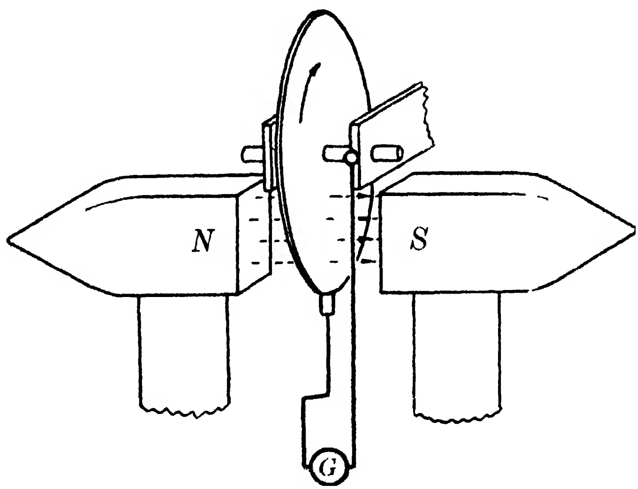


Fig. 29. Faraday's disc

is not changing. In this experiment the configuration of the stationary part of the circuit is immaterial.

If the disc is small enough to be placed between the poles, with its plane parallel to the field, no effect is observed, but a current is produced if two sliding contacts, connected to a galvanometer, press against opposite surfaces of the disc. We thus conclude that, for an e.m.f. to be induced, the conductor must move *across* the field. The e.m.f. is found to be proportional to the speed of the disc, and also to the density of the field. It is shown later that this type of electro-magnetic induction is a necessary consequence of the fact that electric charges, moving in a magnetic field, experience forces per-

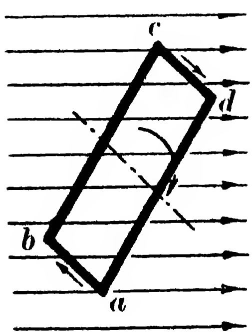
pendicular both to the field and their direction of motion, a fact that is observed directly in the cathode-ray oscillograph.

We recognize in cases (B) and (C) the same mechanism, that of a conductor moving in an unchanging magnetic field. Now since all motion is relative we can take the conductor as being at rest and the magnetic field as moving, but in order to give this statement physical meaning we define the *relative motion between conductor and field, as being that between the conductor and the current system to which the field is due*. This definition of moving fields is of great importance.

In case (B) (that of the motion of a rigid coil in an unchanging field) we have stated that the e.m.f. is proportional to the rate of change of linking flux, and that its direction is given by Lenz's Law. These facts enable us to determine the direction of the e.m.f. induced when a conductor moves across a field.

4. Direction of e.m.f. in a conductor moving in a magnetic field.

Consider the rotating coil of case (B) (Fig. 30, *a*). Let the coil be rectangular, and let it rotate about an axis perpendicular to the field. The sides *bc* and *da* do not cut *across* the lines of force, so we assume that no e.m.f. is induced in them. The total e.m.f. induced in the coil must then be the sum of the e.m.f.'s induced in the sides *ab* and *cd*.



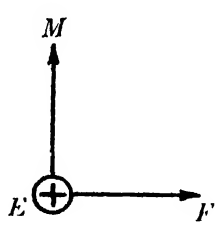


Fig. 30, *a* and *b*. Direction of e.m.f. in moving conductor

As the coil rotates the magnetic flux linking it changes, and in the position shown this flux is *decreasing*. Hence by Lenz's Law the e.m.f must be in the direction *a–b–c–d*. In Fig. 30, *b*, the circle with the cross represents the cross-section of the wire *a–b* with the e m.f. (E) downwards into the paper, F the direction of the field, and M the component of the conductor's motion perpendicular to the field. Then it is seen that the vectors (M, F, E) form a *right-handed* set. (Compare E, F, M for *motor action*.) This relation is also given by:

Fleming's right-hand rule for generator action. This is shown in Fig. 30, *c*, in which the digits of the right hand have the same letters assigned to them as in the left-hand rule of Fig. 25.

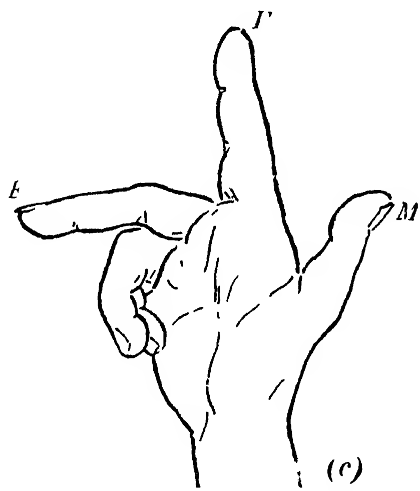


Fig. 30, *c* Fleming's right hand rule for generator action

5. Summary. The essential facts discovered by Oersted and Faraday.

It would naturally be very satisfying if the two methods of inducing an e.m.f. could be shown to be particular cases of one general law, but attempts to do this make use of philosophical speculations which are outside the realm of what is physically definable. We must not endow hypothetical concepts, such as the electric and magnetic fields, with physical properties over and above those inherent in their definitions. For instance,

when we say that a conductor "cuts lines of force", we mean only that the conductor moves relatively to the system of currents which gives rise to the "lines of force", and when we say that a magnetic flux *changes* we mean only that the system of currents, to which it is due, is changing.

A school of thought seems to exist which takes the *cutting of lines of force* as fundamental to the induction of every e.m.f., but to account for case (A) (the iron-cored transformer) by this method entails an extremely artificial conception of the lines of force moving from the iron, out into the surrounding space, and so cutting the conductors of the coil as they move, whenever the field in the iron changes. Again, some writers hold that the *change of linking flux* is the fundamental condition, and indeed in many cases this law in its *quantitative* form is of great value, but case (C) (the rotating disc) cannot be satisfactorily explained by this method. The fallacy in all such arguments lies in the acceptance of the *physical reality* of a magnetic field, and a failure to realize that it is merely a postulated intermediary between moving electric charges.

It is best, then, to accept *two distinct methods* whereby an e.m.f. may be induced by magnetic fields:

(A) A stationary circuit is linked by a stationary magnetic field whose magnitude is changing. It is immaterial whether the wires of the circuit are actually situated *in* this magnetic field or not.

This is equivalent to saying that when a magnetic field changes in magnitude, an electric field* exists around any

* An electric field is defined by the relation

$$E = \frac{F}{q},$$

and whenever a stationary charge experiences a force, we say it is in an electric field whose intensity is the force per unit charge.

The "induced" electric field of electro-magnetism thus has the same "physical meaning" as the electro-static field of stationary charges, but its configuration obeys different laws. Since its lines of force (or displacement, if there is a change of dielectric) are closed loops, it clearly does not follow the inverse-square law, neither is there any

closed path which surrounds any part of the field. The e.m.f. around the closed path is equal to the line-integral of this induced electric field. The phenomenon may be called "transformer" or "flux-linking" induction.

(B) A conductor, which may or may not be part of a rigid coil, is *situated in* a magnetic field, and moves through it (i.e. there is relative motion between the conductor and the current circuit or magnet causing the field) in such a way as to cut across the lines of force.

This is equivalent to saying that, if there is relative motion between an observer and a current-circuit or magnet such that the direction of motion is across the lines of force of the magnetic field of the currents, then for that observer an electric field exists which is perpendicular both to the motion and to the magnetic field The phenomenon may be called "motional" or "flux-cutting" induction

The essential facts discovered by Oersted and Faraday

If we choose to think directly in terms of the action of charge upon charge, dispensing with the services of those useful concepts, the electric and magnetic fields we arrive at the following assessment of the two great discoveries discussed in this chapter.

(A) OERSTED's discovery of the magnetic field of the electric current may be interpreted as the discovery that the force between two electric charges depends on their *velocities* as well as on their *positions*. In his original experiment, charges were moving in the conductor and in the atoms of his compass needle. The force on the compass needle was the resultant of the forces between all the charges in the atomic structure of the needle and the wire.

single-valued difference of potential between two points in it. For we may take two points close together on a closed line of force, and clearly the work done in taking unit charge from point to point, along the line of force, will depend, both in sign and magnitude, on the direction in which the charge travels.

(B) FARADAY, when he discovered that a changing magnetic field could produce an electric current, really discovered that a force is experienced by a *stationary* charge when a charge in its vicinity changes its velocity (i.e. accelerates or decelerates). The electrons in the secondary coil of his iron ring had no directed motion until the electrons in the primary coil accelerated or decelerated: they were then set in motion showing that they were acted upon by a force.

Now the fascinating conclusion, in the light of modern knowledge, is that the *flux-cutting* method of electro-magnetic induction is a logical deduction from *Oersted's* discovery, as will be seen from the following reasoning, and the quantitative development of Sections 6, 7, and 8, in Part IV of this chapter.

(1) Magnets attract or repel one another.

(2) Electric currents are attended by magnetic fields. Therefore they are equivalent to magnets and attract or repel one another.

(3) The force on one conductor, carrying current, is considered to be due to the motion of electrons, in its structure, in the magnetic field of the other.

(4) If we move a conductor across a magnetic field, we impose a motion upon all the electric charges in its structure They will therefore experience a force, showing that they are situated in an electric field. In other words, an e.m.f. exists in the wire, and those electrons which are free to move will constitute a current.

At the time of Oersted's discovery, however, it was by no means certain that an electric current *did* consist of a motion of electric charges, so that such reasoning was highly improbable (Rowland's experiment, mentioned on p. 42, was performed in 1876). To these early experimenters the magnetic field must have seemed a very fundamental concept, and as such enabled Faraday to discover, by deductive reasoning, the flux-cutting method of induction *after* his discovery of "transformer action".

The flux-linking or "transformer" induction of an e.m.f. appears to be a phenomenon totally different from the flux-cutting case; in Chapter v the facts of electro-magnetism are

summarized in terms of the following three components of the forces between electric charges:

(*a*) A force in virtue of their positions,

(*b*) A force in virtue of their velocities,

and (*c*) A force in virtue of their accelerations.

The first component (*a*) is the electro-static force, the second component (*b*) accounts for the flux-cutting induction of an e.m.f., while (*c*) is the basis of the flux-linking or transformer case.

PART IV

QUANTITATIVE DEVELOPMENT, BASED ON FARADAY'S LAW OF ELECTRO-MAGNETIC INDUCTION

1. Introduction.

Any development of the quantitative relations between electricity and magnetism must be based upon fundamental experimental fact, and as our starting-point we may take any one of several such experimental facts. We developed the theory of the electric field from the mutual force between two charges at rest, so we might start here with the laws of force between charges in motion Such laws, however, are too complex to be of much practical use, and nature has provided, in the concept of the magnetic field derived from a knowledge of permanent magnets, an almost perfect intermediary concept in terms of which the laws take on a simple form. Instead, however, of accepting the usual definitions of magnetic quantities which are based upon the hypothetical unit magnetic pole, we shall take as our starting-point Faraday's Law of electro-magnetic induction,* and in developing our theory we shall make use of the qualitative facts discussed in Parts II and III of this chapter.

* We take this law as our starting-point since it leads immediately to a satisfactory (and practical) method of defining unit magnetic flux. A good example of the development of electro-magnetic relations from the force experienced by a charge moving in a magnetic field is found in *The Electro-Magnetic Field*. by H. F. Biggs (Oxford).

2. Definition of unit magnetic flux: the weber and the maxwell.

We have already defined the "physical meaning" of magnetic flux (see p. 77). Let the magnitude of the flux linking a single turn of wire (Fig. 31) be denoted by ϕ. Then by Faraday's Law the e.m.f. induced in the turn when ϕ changes is

$$e = -k\frac{d\phi}{dt},$$

where k is some constant, and the negative sign is the mathematical expression of Lenz's Law.*

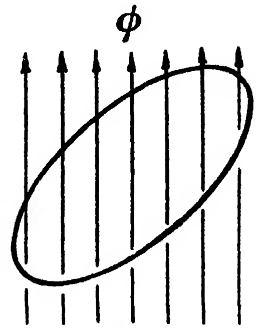


Fig. 31

We proceed to define the unit of magnetic flux by making the constant k equal to unity. Unit flux may then be defined as follows:

DEFINITION OF UNIT FLUX: Unit magnetic flux is that flux which, linking a circuit of one turn, induces unit e.m.f. when it collapses to zero at a uniform rate in unit time.

The *practical* unit, corresponding to the volt and the second, is the WEBER. This name was adopted by the I.E.C. in 1933; previous to this it was usually called a *volt-second*.

The c.g.s. electro-magnetic unit, still used extensively in theory and practice, is the MAXWELL, or *ab-volt-second*, which is often called a *line*.

Since one volt is equal to 10^8 ab-volts, it follows that

$$1 \text{ weber} = 10^8 \text{ maxwells,}$$

or

$$1 \text{ maxwell} = 10^{-8} \text{ weber.}$$

If a coil of N turns is substituted for the single turn of Fig. 31, then, provided the flux ϕ links with each of these turns, the e.m.f. in each turn will be the same, and the total e.m.f. induced in the coil will be equal to the sum of the e.m.f.'s

* The direction of an e.m.f. in the circuit may be denoted by the direction of a vector, normal to the plane of the circuit, the directions of the e.m.f. and this vector being connected by the usual right-handed screw rule. Lenz's Law states that this e.m.f.-axis vector is opposed to the vector denoting $d\phi/dt$. Hence the negative sign.

in the individual turns. Thus the law of induction, for this special case, becomes

$$\oint \mathbf{E} \cdot \mathbf{dl} = e = -N\frac{d\phi}{dt} \qquad (1)^*$$

3. The torque on a coil carrying current, situated in a magnetic field.

Let θ (Fig. 32) be the angle between the plane of the coil and an equipotential (normal) surface of the field. θ is also the angle between the magnetic axis of the coil and the direction of the field.

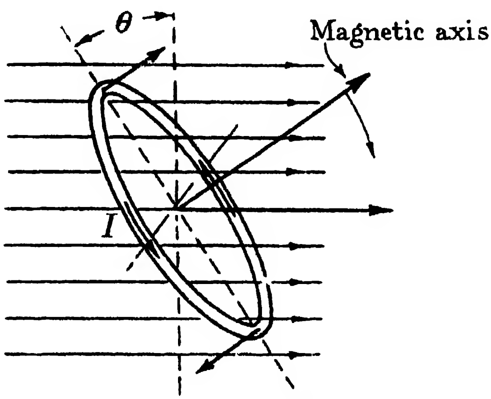


Fig. 32 Coil in a magnetic field

Let the coil carry a steady current I, and let ϕ be the total flux linking the coil. We can consider ϕ to be the sum of the fluxes due to the current I and due to the external field in which the coil is situated. That part due to the current I will be independent of θ, but that due to the external field will depend on this angle.

* This law is sometimes put

$-e =$ rate of change of flux-linkages (i e., of flux-turns),

but if we express this mathematically we get

$$-e = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} + \phi\frac{dN}{dt},$$

which result is *not* consistent with experimental fact. The student is therefore warned against this method of stating the law.

That a torque is experienced by the coil is readily seen by applying the left-hand rule. Now let the coil turn through a small angle $\delta\theta$, in time δt, under the action of this torque, and let the corresponding change in the linking flux be $\delta\phi$. Then an e.m.f. is induced in the coil of value

$$e = -\frac{\delta\phi}{\delta t}.$$

The direction of the torque is such as to turn the coil in the direction which will make ϕ increase (i.e. the magnetic axis of the coil will turn towards the magnetic axis of the external field), so that, by Lenz's Law, the e.m.f. e will oppose the flow of the current I. Hence in order to maintain the current at its steady value, it is necessary to apply an additional e.m.f. in the circuit which is equal and opposite to e, so that additional energy must be supplied to the coil at the rate

$$P = (-e)\,I,$$

or of total amount

$$\delta W = (-e)\,I\,\delta t = I\,\delta\phi.$$

This energy is converted into mechanical work* of amount

$$T\,\delta\theta,$$

where T is the torque on the coil. Hence we have, in the limit,

$$T = I\frac{d\phi}{d\theta}, \tag{2}$$

or, if the coil has N concentrated turns,

$$T = IN\frac{d\phi}{d\theta}. \tag{3}$$

From 2(3) it is clear that the torque on the coil is zero both when the linking flux is a maximum and when it is a minimum. We adopt the convention that flux passing through the coil in the direction of its own magnetic axis is positive, and that flux passing through in the opposite direction is negative. By the use of the motor-action rule it is seen that the equilibrium is stable only when ϕ is a maximum, i.e. when the flux due to

* We have tacitly assumed that none of this energy is used in any other way. For a more rigid proof, see Chapter III, Section 27.

the coil adds to that from the external field. We therefore deduce the following rule:

> When a coil carrying current is situated in a superposed magnetic field and is free to turn about a fixed axis, it will turn in such a direction as to align its magnetic axis, by the shortest route, with the axis of the superposed field.

EXERCISE FOR THE STUDENT

A coil of N turns is so constrained that it can move only without rotation. It is situated in a non-uniform magnetic field, and carries a current I. If ϕ is the flux linking the coil, and x denotes its position in the direction in which it tends to move, show that the force on the coil is given by

$$F = IN\frac{d\phi}{dx}, \tag{4}$$

and that stable equilibrium is again attained when the linking flux (taken as positive in the direction of the coil's magnetic axis) is a positive maximum.

We may therefore state the following simple rule:

> When a coil carrying current is situated in an external magnetic field, it will tend to move by the shortest route into that position in which the linking flux is a maximum.

Illustrative example. Fig 33 shows the mechanism of a moving-coil galvanometer. The needle is fixed to a rectangular

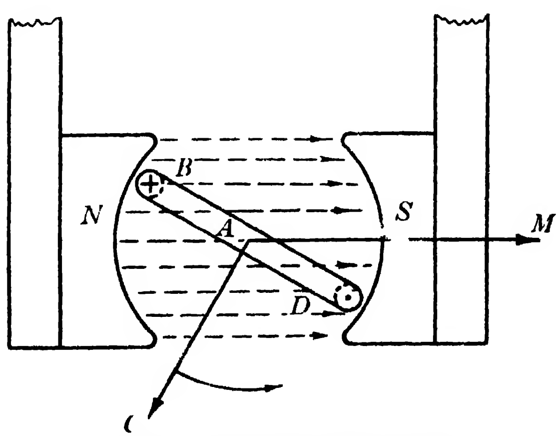


Fig. 33. Moving-coil galvanometer

coil, whose plane is perpendicular to the paper, and which is free to rotate about the axis A in a magnetic field normal to this axis.

If the direction of current in the coil is downwards in side B, and upwards in side D, the magnetic axis of the coil is C, and the magnetic axis of the external field (due to the permanent magnet) is M.

Then by the above rule the axis C will tend to align itself, by the shortest route, with the axis M. Therefore the coil will turn in a counter-clockwise direction.

This result should be verified by applying the left-hand rule.

4. The density of a magnetic field: the flux-density, B.

The density of a magnetic field at a point is equal to the amount of flux passing through unit area of an equipotential surface at the point. In a uniform field the flux-density is constant at all points in the field, but in a non-uniform field it varies from point to point, and in such a case the density at a point must be calculated on the assumption that the field is uniform, having the same density over unit area as it has at the point considered. That is, if $\delta\phi$ is the flux passing through a small area δA of an equipotential surface at the point, the flux-density at that point is given by

$$B = \frac{\delta\phi}{\delta A}, \tag{5}$$

so that the flux through any area of an equipotential surface is equal to the surface-integral of the flux-density:

$$\phi = \iint B\,dA, \tag{6}$$

where the integration is performed over the whole of the surface.

e flux-density B is a measure of the *strength* of the field,
rce experienced by a given charge
velocity through the field, as will be

Units of flux-density. The practical (m.k.s.) unit of B is *one weber per square metre,*

and the c.g.s. unit is *one maxwell per square centimetre,* or *one gauss.*

So that, since one weber is equal to 10^8 maxwells,

$$\left.\begin{array}{l}
\text{One weber per sq. metre} = 10^4 \text{ gauss} \\
\text{and} \quad \text{One gauss} = 10^{-4} \text{ w.p.s.m.} \\
\text{A further unit much used in practice is \textit{one maxwell}} \\
\text{\textit{(or line) per square inch.}} \\
\text{One gauss} = 6{\cdot}45 \text{ lines per sq. inch.} \\
\text{\textit{One kilo-line per sq. inch} is equal to 1000 maxwells} \\
\text{per sq. inch, so that} \\
1 \text{ w.p.s.m.} = 64{\cdot}5 \text{ kilo-lines per sq. inch} \\
\text{or} \quad 100 \text{ kilo-lines per sq. inch} = 1{\cdot}55 \text{ w.p.s.m.}
\end{array}\right\} \quad (7)$$

5. The magnetic moment of a coil, carrying current.

Let a plane coil of N concentrated turns, carrying a current I, be situated in a uniform magnetic field of density B (*Fig. 34*), and let θ be the angle between the field and the plane of the coil.

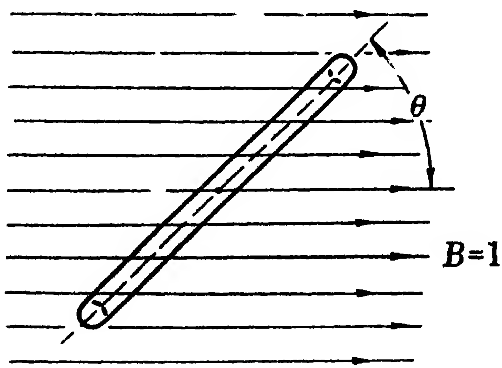


Fig. 34. Coil in a magnetic field

If A is the area of the coil, the flux linking it will be

$$\phi = BA \sin \theta,$$

so that the torque on the coil (equation 2(3)) is

$$T = IN\frac{d\phi}{d\theta} = INAB\cos\theta.$$

The maximum value of this torque is when $\theta = 0$, and is given by

$$T_{\max} = (INA)\,B.$$

The quantity in brackets is called the *magnetic moment* of the coil, T_m, so that:

$$T_m = INA. \tag{8}$$

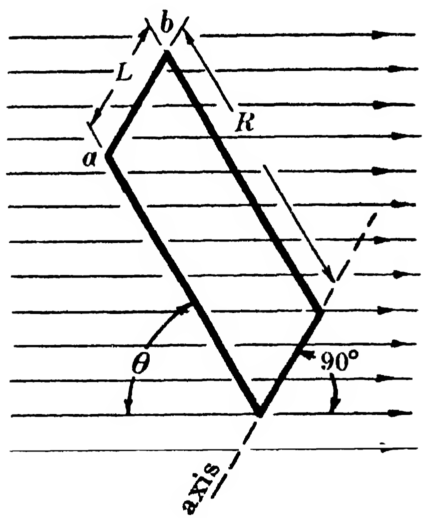
The magnetic moment of a coil is thus numerically equal to the maximum torque the coil will experience when situated in a magnetic field of unit density.

It is also equal to the product of the current-turns and the area.

6. The force on a straight conductor, carrying current in a magnetic field.

Let a rectangular loop (Fig. 35, *a*) of sides L and R be pivoted about an axis coincident with one of the sides L, and situated in a uniform magnetic field of density B, perpendicular to the sides L of the loop. Let the plane of the loop make an angle θ with the direction of the field. Then the flux linking the coil is

$$\phi = BLR\sin\theta.$$



(*a*)

Fig. 35 (*a*). Force on a conductor in a magnetic field

The torque on the coil, when carrying a current I, will be

$$T = I\frac{d\phi}{d\theta} = IBLR\cos\theta.$$

Consider the case when the plane of the loop is parallel to the field, i.e. $\theta = 0$. In this position there is no force on the sides R and the torque is due entirely to the force on the side a–b. Let this force be F, then

$$T = BLIR = FR,$$

so that the force on the side a–b is

$$F = BLI \tag{9}$$

$$(F = BLI \times 10^{-1}).$$

Equation 2(9) therefore gives the force experienced by a straight wire of length L, carrying a current I, when situated perpendicular to a magnetic field of density B.

The general case. Now let the loop be shaped as in Fig. 35, b, the side a–b making an angle α with the sides whose mean length is R. The flux linking the coil is

$$\phi = BLR\sin\alpha\sin\theta,$$

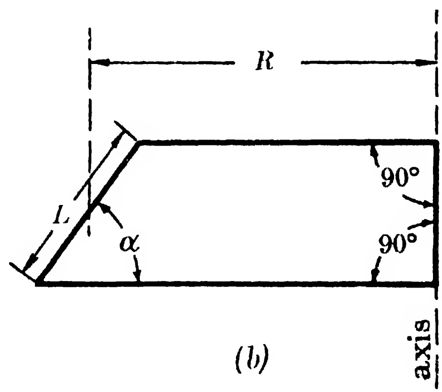


Fig 35 (b)

and it follows from the same reasoning that the force on the conductor a–b, inclined at an angle α to the direction of the field, is

$$F = BLI\sin\alpha, \tag{9a}$$

which is thus the general expression for the force on a conductor carrying current, in a uniform magnetic field.

7. The force on an electric charge, moving in a magnetic field.

A single charge q, moving with velocity v, is equivalent (equation 1(44)) to a current I flowing in a short conductor of length δl, where $qv = I\,\delta l$.

Hence if a charge is moving in a direction making an angle α with the field through which it moves, it should, according to equation 2(9a), experience a force

$$F = B\,\delta l\,I\sin\alpha = Bqv\sin\alpha \quad \text{or} \quad \mathbf{F} = q(\mathbf{v}\times\mathbf{B}),^{*} \qquad (10)$$

perpendicular to its motion and to the field, the mutual directions of the vectors being given, for a positive charge, by the left-hand rule in which the current vector is replaced by the direction of motion of the charge. If the charge is negative, the direction of the force will be reversed. This relation is verified by experiments on cathode rays (streams of electrons) in magnetic fields

The total force on a moving charge Let the charge be moving with velocity v relative to an observer to whom an electric field E and a magnetic field B exist. Then, combining the results of 1(3) and 2(10), we get for the force experienced by the charge q

$$\mathbf{F} = q\{\mathbf{E} + (\mathbf{v}\times\mathbf{B})\}. \qquad (10a)$$

We should notice that this equation assumes that the force on the charge due to the electric field E is unaffected by the motion of the charge Our definition of E presupposes a *stationary* charge, by which the electric field is measured, and this extension of the meaning of E beyond that inherent in its definition is justified only by the fact that it leads to calculated effects which are consistent with experimental fact.

8. The flux-cutting induction of an e.m.f.

We have already seen that an e.m.f. appears to be induced in a conductor when it is moved across a magnetic field. We shall now make use of equation 2(10) to derive the quantitative law governing this case.

* For an explanation of this vector notation, see p. 98.

Consider a straight conductor of length L, perpendicular to a uniform magnetic field of flux-density B (Fig. 36). Let this conductor move, in a direction perpendicular to the field and to its length, with velocity v. Consider any positive charge q located inside the structure of the conductor. Then by 2(10) the motion of this charge results in its experiencing a force (upwards along the wire in the case shown)

$$F = Bqv.$$

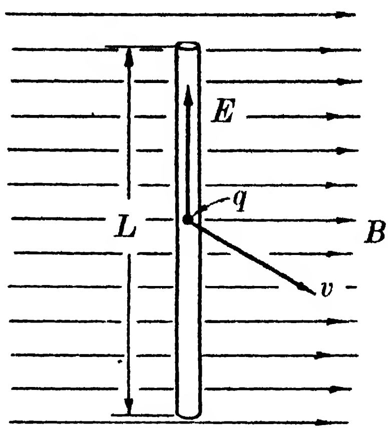


Fig. 36. Flux-cutting induction of e.m.f.

This means that the conductor is subject, throughout its interior, to an electromagnetic *force per unit charge* which we denote, as on p 50, by F_i, and call the *electromagnetic force*, where

$$F_i = Bv, \tag{11}$$

or, if the direction of motion of the conductor makes an angle α with B,

$$F_i = Bv \sin \alpha, \tag{11 a}$$

so that, by 1(37 e), an e.m.f exists between the ends of the conductor given by

$$e = F_i L = BLv. \tag{12}$$

(If the motion is not perpendicular to the field, v may be taken as the component of velocity perpendicular to the field.)

[If B is in *gauss*, L in *centimetres*, and v in *cm. per sec.*,

$$e = B \times 10^{-4} L \times 10^{-2} v \times 10^{-2} = BLv \times 10^{-8} \text{ volts.}]$$

Now in 2(12), the product Lv is equal to the area swept through by the conductor in one second, and the total magnetic flux over this area is equal to BLv. It is possible to show, by considering the proper components of the various vector quantities involved, that this result is also true for the general case, where the field, conductor, and direction of motion are not mutually perpendicular. We are thus led to a rule which sometimes proves extremely useful, especially in calculating

the *mean* value of the e.m.f. in a conductor which moves through a non-uniform field:

> When a conductor moves in a magnetic field, an e.m.f. is induced whose value is equal to the rate at which magnetic flux is cut by the conductor.*

Thus the mean value of the e.m.f. induced in a conductor on the rotating armature of a generator, is equal to the flux which the conductor cuts in one second.

Induction by moving magnetic fields. If the conductor in the above discussion is stationary, and is situated in a magnetic field whose source (current-circuit or magnet) is moving, an e.m.f. will again be induced in the conductor of value given by 2(12), but the mutual directions of e, B, and v_m are now given by the *left-hand* rule. The velocity v_m is now the velocity of the magnetic field relative to the conductor, calculated on the assumption that the field partakes in the motion of its source, whereas v in 2(12) is the velocity of the conductor relative to the field so that

$$v_m = -v.$$

An extremely valuable "short-hand" in which to express the flux-cutting law is found in the "cross-product" notation of vector analysis. Let $\mathbf{B}$ and $\mathbf{v}$ be two vectors making an angle α with each other. Then the expression

$$\mathbf{B} \times \mathbf{V}$$

* This rule may also be used to calculate the e.m.f. in Case A (that of a coil linked by a changing field). For instance, during an interval of time δt let the flux linking one turn of a transformer winding decrease by $\delta\phi$. If we imagine that this amount of flux has not actually disappeared, but has *cut across* the turn and gone out somewhere into the region exterior to the coil, we see that the rate of this hypothetical "flux-cutting" is equal to $\delta\phi/\delta t$, which is the value of the e.m.f. in the turn given by the flux-linking law, 2(1), and, further, the direction of the e.m.f. deduced by this artifice is correct. Although this method of calculation gives the correct result (see, for example, "The Calculation of Induced Voltages in Metallic Conductors", by *H. B. Dwight, Tr. A.I.E.E.* April p. 447) the student would be well advised to avoid the temptation to endow the idea with physical reality. In Chapter v we shall see that the flux-cutting rule is also applicable, if used with careful restrictions, in calculating the e.m.f. induced by an electro-magnetic wave, but that to endow the fields of a wave with physical motion is logically inadmissible.

is called the *vector-product* of the vectors, and is defined as being a third vector, perpendicular to both **B** and **v**, of magnitude $Bv \sin \alpha$, and in such a direction that if we turn the vector **B** towards the vector **v**, and at the same time progress in the direction of motion of a right-handed screw, we move in the direction of this third vector Clearly

$$\mathbf{B} \times \mathbf{v} = -\mathbf{v} \times \mathbf{B}$$

Applying this notation to electro magnetic induction by flux-cutting, we get

(A) A conductor moving with velocity v through a stationary magnetic field **B** is subjected to an electro-magnetic force

$$\mathbf{F} = \mathbf{v} \times \mathbf{B} \qquad (12b)$$

(B) A stationary conductor, situated in a magnetic field which is moving with velocity **v** is subjected to an electric field given by

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} \qquad (12b)$$

The velocity **v** in 2(12*b*) is the velocity of the *source* of the magnetic field **B** relative to the conductor. The relation is true only if all parts of the magnet or current-circuit are moving with the same velocity That is if there is no rotation. In the case of a rotating magnet. the relation applies to each element of the magnet, and in consequence it is difficult to obtain the exact value of **E** in such a case. Fortunately, however, the *total* e.m f in a closed conducting circuit can always be obtained by assuming that the field of a magnet has both the translatory and rotary motion of the magnet

9. The use of the flux-linking and flux-cutting laws for electro-magnetic induction

We repeat the statement of the two distinct cases of electro-magnetic induction:

(1) An electric field exists around any closed path which links a changing magnetic field. The total e.m.f. around the path is given by $e = -d\phi/dt$.

(2) Moving through a magnetic field, we experience an electric field which is perpendicular, both to the field and to

the direction of motion, given by $E = Bv \sin \alpha$, where α is the angle between the direction of the magnetic field and the direction of motion, B is the flux-density of the field, and v is the velocity of the observer relative to the current system to which the magnetic field is due. In the case of a conductor situated at right angles to the field, the e.m.f. induced is given by $e = BLv$, where L is the length of the conductor, and v is the component of the conductor's velocity perpendicular to the field.

We now give the following "working rules" for the calculation of induced e.m.f.'s:

(A) *The flux-linking law* ($e = -N\,d\phi/dt$) *should be used* whenever a circuit (of constant or of changing shape, stationary or moving) is linked by a changing field. The sum of both "transformer" and "motional" e.m.f.'s is thus obtained.

(B) *Either law may be used* when a rigid coil moves in any way in an unchanging field.

(C) *The flux-cutting law* ($e = BLv$) *should be used* for cases of the motion of conductors through constant magnetic fields, where the circuit consists of two or more sections of constant shape, between which there is relative motion, the sections being connected by sliding contacts. As examples we may cite Faraday's disc (Fig. 29), and the homopolar generator of Ex. 2 of this chapter. In these two cases the configuration of the circuit remains unchanged by the rotation of disc or cylinder. A case where the circuit-configuration is altered by the motion is that of Ex. 20 of this chapter.

The rigorously correct method of calculating the e.m.f. (which gives the e.m.f. induced in each section of the circuit) is to treat each component of the magnetic field as sharing the motion of its source (see Chapter III, Sections 2 and 3), for this method recognizes the fundamental cause of motional induction: relative motion between matter and current-circuits (or magnets). In

some cases, such as that of the homopolar generator, the total e.m.f. in the circuit may be calculated on the assumption that the magnetic field is stationary with respect to any one section of the conducting circuit, but in Ex. 20 of this chapter a case is given where the rigorous method is necessary.

(D) *Both laws must be used* whenever we have a combination of cases (A) and (C).

10. E.m.f. and terminal p.d. in a closed circuit.

A. *The circuit of an electro-magnetic generator*

In generators (machines in which mechanical energy is converted into electrical energy) there is usually relative motion between the conductors of the generator itself and those which form the external stationary circuit. The relation between the directions of the current, induced e.m.f., and terminal p.d. in the different parts of the circuit is not immediately obvious, and is worth examination.

Consider the simplified case of Fig. 37. A straight wire AB moves from left to right, making contact with two stationary copper bars by means of sliding contacts at its ends. Let AB move through a magnetic field which is perpendicular to the paper and downwards (i.e. away from the observer).

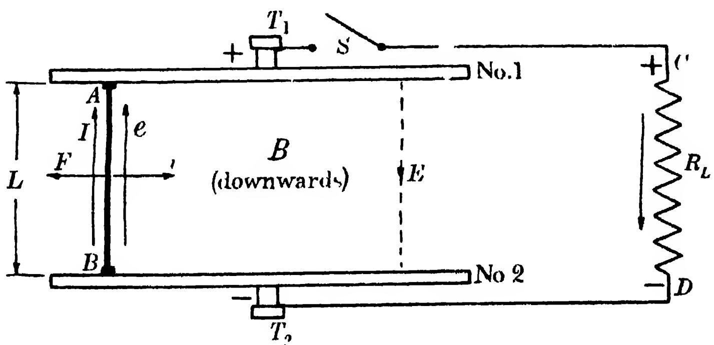


Fig. 37 Principle of generator

Moving with AB, we are situated in an electric field which, by the right-hand rule, acts from B to A, and is given by

$E = Bv$, so that if this field is not affected by anything that happens in the conductor, we measure an e.m.f. from B to A of value $e = BLv$.

Now the mobile electrons in AB will move under the influence of this field towards B until the net electric field inside the wire is reduced to zero. That is, a negative charge is induced at the end B and a positive charge at the end A, the field due to these induced charges exactly cancelling E; but the stationary bars, to which terminals T_1 and T_2 are attached, are in contact with the ends A and B, so that a p.d. equal to e now exists between the two stationary bars, *bar No. 1 being positive to bar No. 2*

To an observer stationary with respect to the bars, the field E in the wire AB does not exist (it exists only for an observer moving with AB) and he is conscious only of the p.d., $V = e$, due to the induced charges on the bars. The induced e.m.f. in the moving conductor, e, thus causes a potential difference V, equal in magnitude to e, between the parallel bars. We say that T_1 is the positive terminal, and T_2 the negative terminal, of the generator.

Now let a piece of apparatus which absorbs electrical energy, CD (for example, an incandescent lamp of resistance R_L ohms), be connected across T_1 and T_2, through a switch at T_1 which is at first left open.

While the switch S is open, the whole of the stationary circuit from T_2 to the switch will be at the same (negative) potential as T_2, so that the p.d. V (the "open-circuit" p.d. of the generator) now appears across the switch S.

Now let the switch be closed: the immediate result is a rush of electrons across it in the direction from C to T_1 (i.e. current is said to flow from T_1 to C). This motion of electrons immediately spreads through the bar No. 1 to the end A of the moving wire. Going back to our observation post inside AB, we are now conscious of an electric field from B to A, and any free electrons will start moving towards B. This motion of electrons is very quickly established around the whole circuit, and a steady current will flow so long as the bar moves with constant velocity.

Since the apparatus CD has resistance R_L, there will be a fall of potential from C to D while the current is flowing. Conditions now appear to the stationary observer thus:

(1) The current I flows from + to − through CD (i.e. in the direction of *potential drop*).

(2) The current I flows from − to + through the generator (i.e. in the direction of *potential rise*).

In conventional circuit diagrams we show a generator by means of a circle and make no attempt to depict the moving conductors. Fig. 38 shows the circuit of Fig. 37 in this con-

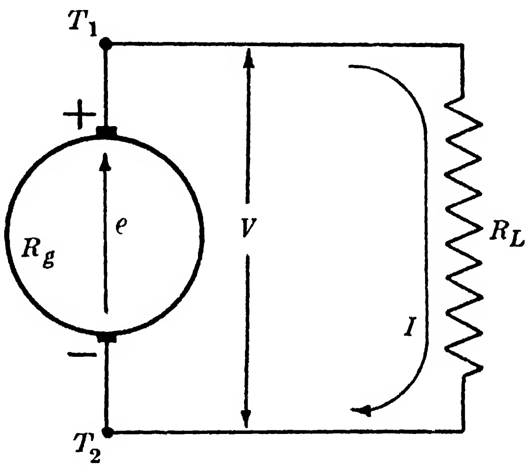


Fig 38 Conventional diagram of generator

ventional manner, and the e m.f e in the moving conductors is denoted by an arrow from the *negative* to the *positive* terminal, or in the same direction as the current flow

From the standpoint of the stationary observer, the generator appears to pump electrons *against* the electric field between the generator terminals. The potential energy thus given them is given up in some useful form in the absorbing apparatus CD.

The relation between the direction of current flow and the terminal p.d., in electrical apparatus, is thus given by the following rules:

(A) When there is a net absorption of electrical energy in electrical apparatus, the current flows through the

apparatus in the direction of *fall* of potential between the terminals.

(B) When there is a net output of electrical energy from electrical apparatus, the current flows through the apparatus in the direction of potential *rise* between the terminals.

The power generated: Kirchhoff's Second Law. The mechanical work which is converted into electrical energy is that which must be done in moving the conductor AB (Fig. 37) against the force it experiences in carrying a current I in the magnetic field B. This force, F, by the principle of conservation of energy (and also by Fleming's left-hand rule) must oppose the motion v, and is given by

$$F = BLI. \qquad 2(9)$$

The mechanical work done per second (power) in overcoming this force is

$$P = Fv = BLvI = eI, \qquad (13)$$

which is therefore the *total electrical* power generated.

The electrical power *absorbed* in the complete circuit must be equal to this generated power, so that (I being constant)

$$P = eI = I^2(R_g + R_L) = I^2R,$$

where R is the total resistance in the circuit. Hence

$$e = I(R_g + R_L) = IR.$$

We have, then, a relation in a closed circuit which is known as *Kirchhoff's Second Law*:*

At any instant, the total electro-motive force in a closed conducting circuit is equal to the product of the current and the resistance of the circuit (IR).

The terminal p.d., V. The potential difference between the terminals, when the current I is flowing, is given by

$$V = IR_L = e - IR_g,\dagger \qquad (14)$$

* Kirchhoff's *First Law* of current circuits is that the algebraic sum of the currents which meet at any point is zero.

† An interesting point is the difference between the field intensities measured by different observers. An observer situated in the moving wire AB is conscious of a field from B to A of value IR_g/L only, where R_g is the resistance of the moving wire. That is, the only energy con-

from which we see that there are two limiting cases:

(A) On open-circuit: $I = 0$ and $V = e$.

(B) On short-circuit: $R_L = V = 0$ and $I = \dfrac{e}{R_g}$.

Further, the *output* of electrical energy to the load is

$$P_0 = VI = eI - I^2R_g \tag{15}$$
$$= \text{(total electrical power generated)}$$
$$- \text{(power wasted as heat in the generator)}.$$

B. *The circuit of a transformer winding*

Now consider the case of the e.m.f. induced by a changing magnetic field. Let the dotted loop in Fig. 39(*a*) represent any closed path linking a changing magnetic flux ϕ. Then there appears an electric field around this path of total e.m.f. given by

$$e = \oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d\phi}{dt},$$

in the direction shown. Let the dotted path be almost completely filled by a conducting wire (Fig. 39(*b*)), whose ends

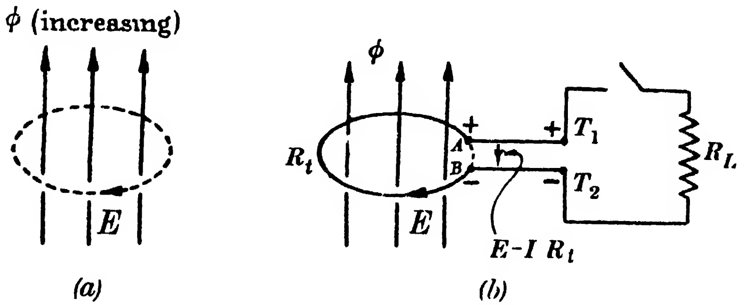


Fig. 39. Transformer winding

version *inside* the wire is that of I^2R_g watts into heat. While the current is flowing, the separation of charges inside AB is such as to cause a p.d. (from A to B) of $e - IR_g$ volts. Superposed upon this is the induced e.m.f. e, from B to A, so that the net drop in the wire is IR_g from B to A. The stationary observer, however, is conscious only of the p.d. due to the separated charges, i.e. he measures a terminal p.d. of $V = e - IR_g$ volts between T_1 and T_2. The actual conversion of mechanical to electrical energy appears to take place in the electromagnetic field *outside* the moving conductor. See also Chapter v, Section 7, and *Engineering*, CXLV (June 10, 1938), p. 648.

A and B are connected to terminals T_1 and T_2. Inside the conductor, the induced field E causes a separation of positive and negative charges (electrons moving to the terminal T_2) whose field will exactly cancel E in the wire. The induced e.m.f., e, is apparently concentrated between the ends A and B, and the terminals T_1 and T_2, the field between these terminals being actually due to the displaced charges in the terminal connections.

When current flows, due to the connection of a load resistance R_L to the terminals, it will flow (as in the generator) from + to − through R_L and from − to + through the coil. The voltage drop in the coil itself is equal to IR_t (where R_t is the resistance of the coil), which must therefore be the difference between the induced e.m.f. ($e = -d\phi/dt$, where ϕ is now due in part to I), and the line-integral around the circuit, from terminal to terminal, of the electro-static field of the charges on the terminals. This electro-static field produces the terminal p.d., which is thus seen to be equal to $(e - IR_t)$.

The circuit considered corresponds to the secondary of a transformer. The reader may now analyse for himself the conditions in: (a) an inductance coil when connected to an alternating p.d., and (b) a complete transformer, consisting of primary and secondary windings.

C. *The circuit of an electric motor*

The simplified circuit of Fig. 37 may be used to illustrate the fundamentals of motor action by replacing R_L by a battery of e.m.f. e_b, and allowing the bar AB to move under the influence of the force $F = BLI$ which it experiences due to its carrying a current I in the magnetic field B (Fig. 40a).

Let the total resistance of the circuit ("motor" and battery) be $R = R_m + R_b$. Then if the bar is not allowed to move, a current will flow given by

$$I_s = \frac{e_b}{R}.$$

This is called the *stand-still current* of the motor. Now suppose the wire is allowed to move under the force BLI_s and to do useful mechanical work in so moving.

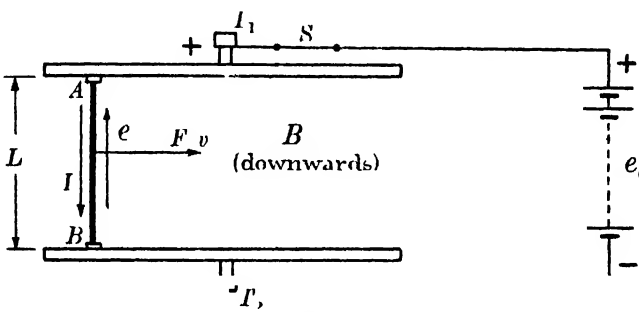
Let its velocity at any instant be v. Then owing to its motion in the field B an e m.f. $e = BLv$ is induced from B to A which, as before, is opposed by the electro-static field between the terminals. The current flows in the direction of this electro-static field, and against e, which is therefore called the *back e.m.f.* of the motor

Kirchhoff's Second Law now gives for the circuit (Fig. 40*b*):

$$e_b - e = I(R_m + R_b) = IR,$$

so that

$$I = \frac{e_b - e}{R}.$$



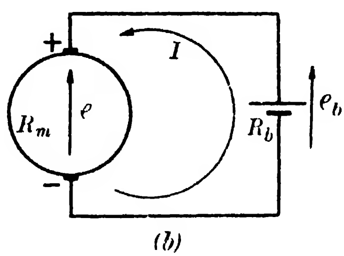


Fig 40. Principle of motor

Since the back e.m.f. e is proportional to the speed v, it follows that the current I, and the driving force F, will decrease so long as v increases. This process will terminate when the driving force F has decreased to a value equal to the force opposing motion, at which point I and v become constant. Also, since e is proportional to B, it follows that, the stronger the magnetic field, the lower will be the final steady speed for a given opposing force

The terminal p.d. of the motor is equal to that of the battery, assuming the connecting leads to have negligible resistance. That is,

$$V = e_b - IR_b = e + IR_m. \tag{16}$$

The mechanical power developed by the motor is

$$P_m = Fv = BLIv = eI, \tag{17}$$

while the electrical power supplied to the moving bar is

$$P_e = VI = eI + I^2R_m \tag{18}$$

= (mechanical power generated)

+ (power wasted as heat in the motor).

11. The rates of energy conversion and energy transfer in a circuit.

(A) *The rate of electro-magnetic conversion of energy.* Referring to equations 2(13) and 2(17) above, we see that in each case energy is converted by electro-magnetic action, from one form to another, at the rate $P = eI$. We may express this important relation in a general law.

> The rate at which electro-magnetic conversion of energy takes place, in electrical apparatus, is equal at any instant to the product of the generated electro-motive force of the apparatus, and the current flowing.

(B) *The rate of energy transfer, from one part of a circuit to another.* Equations 2(15) and 2(18) give, respectively, the rate at which energy is given out by the generator, and taken in by the motor. In each case it is given by $P = VI$, where V is the terminal p.d. If now the generator is connected to the motor, the whole forming a closed circuit, $P = VI$ gives the rate at which energy is transferred from the generator to the motor. We may generalize this statement as follows:

> The rate at which electrical energy is exchanged, between two parts of a closed circuit, is equal to the product of the current flowing, and the potential difference between the two points which divide the circuit into the two parts.

EXAMPLES, CHAPTER II

1. A metal disc, of radius 15 cm. and rotating at 1000 r.p.m., is situated in a uniform magnetic field of density 1000 gauss, such that the rotational axis of the disc is coincident with the direction of the field. Calculate the e.m.f. induced in the disc between centre and rim. In what direction does it act?

Solution. In order to measure the e.m.f., a voltmeter must be connected between the disc's centre and rim by means of sliding contacts. The e.m.f. measured is therefore that induced in all parts of this circuit, but by our theory the whole of the e.m.f. is induced in the disc, provided that the voltmeter circuit is stationary with respect to the magnetic field. It is clearly an example of Case C, Section 9 (p. 100).

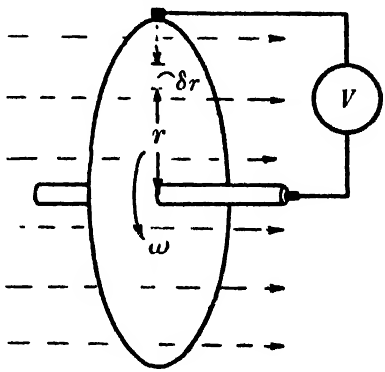


Fig. 41. Rotating disc

Consider the e.m.f. induced in a radial element δr situated at a distance r from the centre of the disc (Fig. 41). The velocity of this element relative to the voltmeter circuit is ωr, where ω is the angular velocity of the disc. If the flux-density of the field is B, the e.m.f. induced in the element is

$$\delta e = B\omega r\,\delta r \qquad \text{(by 2(12))}$$

so that

$$e = \int_0^R de = \tfrac{1}{2}B\omega R^2,$$

and by the right-hand rule for generator action the direction of the e.m.f. is from centre to rim.

Now $R = 0{\cdot}15$ metre,

$$\omega = 2\pi\frac{1000}{60} = 104{\cdot}6 \text{ radians per sec.},$$

$$B = 1000 \text{ gauss} = 0{\cdot}1 \text{ Wb/m}^2,$$

hence $E = 0{\cdot}5 \times 0{\cdot}1 \times 104{\cdot}6 \times (0{\cdot}15)^2 = 0{\cdot}118$ volt.

(*Note.* This problem may be solved without the use of the calculus by taking the e.m.f. as equal to the flux cut per second by a radius of the disc. The method given is more complete in that it gives the induced electric field intensity at every point on the radius.)

2. Fig. 42 shows a cross-section of a homopolar generator. B is a copper cylinder which rotates in a radial field between two cylindrical pole-pieces A and C, of which A is the north pole. The circuit is completed by means of two fixed brushes sliding on the cylinder, one at each of its ends. The mean flux-density of the field over the thickness of the cylinder wall is 50,000 maxwells (lines) per sq. inch, the mean

radius of the cylinder is 5 in., its axial length 20 in., and it rotates at 2000 r.p.m. Find the magnitude and direction of the e.m.f. which appears between the brushes.

Solution. This again is an example of Case C, p. 100. From the data it is clear that the e.m.f. measured by the voltmeter is induced entirely in the rotating cylinder.

The peripheral speed of the cylinder, relative to the fixed voltmeter, is

$$v = \frac{2000}{60} \times 2\pi \times 5 \times 0{\cdot}0254 = 26{\cdot}6 \text{ metres per sec.},$$

$$L = 20 \times 0{\cdot}0254 = 0{\cdot}508 \text{ metre},$$

$$B = \frac{50{,}000}{(2{\cdot}54)^2} \text{ gauss} = 0{\cdot}775 \text{ weber per sq. metre};$$

hence $$e = BLv = 0{\cdot}775 \times 0{\cdot}508 \times 26{\cdot}6 = 10{\cdot}47 \text{ volts}.$$

The direction of this e.m.f. is, in the rotating cylinder, towards the reader.

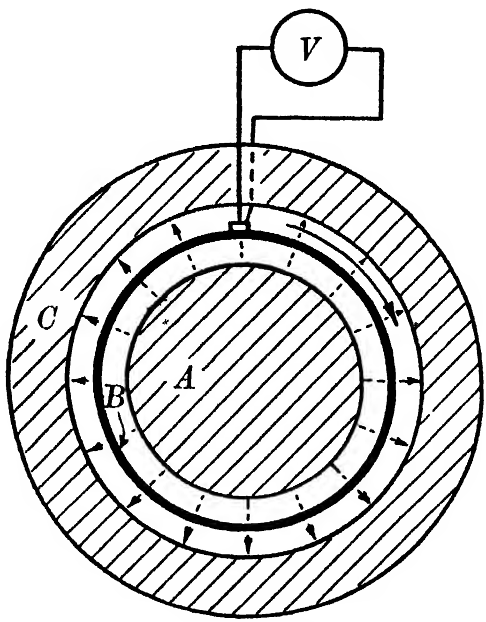


Fig. 42. Homopolar generator

3. In the last example, suppose that it is possible to make the whole of the external circuit (that is, that part extending from brush to brush through the voltmeter) rotate in a clockwise direction at a speed of 1000 r.p.m. Find the magnitude and direction of the e.m.f. in the circuit

(a) with the cylinder at rest,

(b) when the cylinder rotates clockwise at 2000 r.p.m.,

(c) when the cylinder rotates counter-clockwise at 1000 r.p.m.

Solution. As stated in Case C, Section 9, of this chapter, the relative velocity of the two parts of the circuit may be used in calculating the e.m.f.

(a) The cylinder rotates at 1000 r.p.m. in a counter-clockwise direction with respect to the voltmeter. Hence

$$v = 13{\cdot}3 \text{ metres per sec.}$$

and $e = 5{\cdot}23$ volts in a direction *opposite* to that in Example 2.

(b) The cylinder now rotates at 1000 r.p.m. in a clockwise direction with respect to the voltmeter. Hence $e = 5{\cdot}23$ volts in the *same* direction as in Example 2.

(c) The cylinder rotates counter-clockwise at 2000 r.p.m. with respect to the voltmeter, so that $e = 10{\cdot}47$ volts, in a direction opposite to that of Example 2.

4. The copper cylinder and external circuit of the same generator remain stationary, but the cylindrical pole-pieces are arranged to rotate about their axis, either with the same or different angular velocities. Do you expect such rotation of the magnet poles to result in any induced e.m.f. in the voltmeter circuit?

5. A square coil of side 10 cm. rotates at 1000 r.p.m. about an axis passing through the centres of two opposite sides, in a uniform unchanging field of density 1000 gauss, perpendicular to its axis of rotation. Calculate the e.m.f. generated in the coil.

Solution. This is an example of Case B, p. 100, and we shall apply both rules in turn.

(a) *By the flux-linking law.* Let θ (Fig. 43) denote the position of the coil at time t, measured from the instant when $\theta = 0$. Then $\theta = \omega t$, where ω is the angular velocity of the coil.

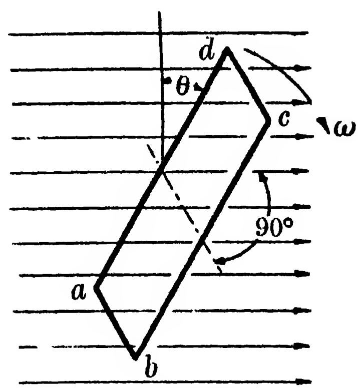


Fig. 43. Rotating coil

Let the side of the coil be of length d, and let B be the flux-density of the field.

Then at time t the flux linking the coil is
$$\phi = d^2 B \cos \omega t,$$
and the e.m.f.
$$e = -\frac{d\phi}{dt} = d^2 \omega B \sin \omega t.$$
The e.m.f. thus alternates in a simple-harmonic manner.

(b) *By the flux-cutting law.* The sides b–c and d–a of the coil have no component of motion perpendicular to the field, so that no e.m.f.'s are induced in them. The total e.m.f. in the coil must therefore be equal to the sum of the e.m.f.'s induced in the sides a–b and c–d, due to their motion in the field.

The speed of each of these sides of the coil is

$$v = \frac{\omega d}{2},*$$

and at time t the direction of this velocity makes an angle θ with the field. The e.m.f. in the side a–b is

$$e_{ab} = BLv \sin\theta = Bd\frac{\omega d}{2}\sin\omega t = \tfrac{1}{2}d^2\omega B\sin\omega t,$$

and the e.m.f. in c–d has the same value, in such a direction as to add to e_{ab} around the coil. Hence the total e.m.f. in the coil is

$$e = d^2\omega B\sin\omega t.$$

(The student should apply both Lenz's Law and Fleming's Right-hand Rule to determine the direction of this e.m.f.)

Inserting numerical values:

$$d = 0{\cdot}1 \text{ metre},$$

$$\omega = \frac{2\pi \,.\, 1000}{60} = 104{\cdot}6 \text{ radians per sec.},$$

$$B = 1000 \text{ gauss} = 0{\cdot}1 \text{ Wb/m}^2;$$

whence $\quad e = 0{\cdot}1046 \sin(104{\cdot}6t)$ volts.

6. A square coil of side d rotates, as in Ex. 5, in an alternating magnetic field whose flux-density at time t is given by

$$B = B_m \sin(pt+\alpha),$$

where t is measured from the instant when $\theta = 0$. If the angular velocity of the coil is ω, find an expression for the induced e.m.f.

Solution. This is an example of Case A, p. 100, so that the flux-linking law is to be used.

At time t, the flux linking the coil is

$$\phi = d^2B\cos\omega t = d^2B_m\sin(pt+\alpha)\cos\omega t.$$

$$e = -\frac{d\phi}{dt} = -d^2pB_m\cos(pt+\alpha)\cos\omega t + d^2\omega B_m\sin(pt+\alpha)\sin\omega t$$

(The first term is a "transformer" e.m.f., the second a "motional" e.m.f.)

$$= -\frac{d^2B_m}{2}[(p+\omega)\cos\{(p+\omega)t+\alpha\} + (p-\omega)\cos\{(p-\omega)t+\alpha\}].$$

* It seems reasonable in this example to agree that the e.m.f.'s in a–b and c–d are equal, provided that the axis of the coil is at rest with respect to the magnet system which sets up the field, as is the case in an ordinary generator. For the purpose of calculation, however, we may take the velocity of a–b relative to c–d as the value of v to be used in the equation: we then account for the total e.m.f. in the coil by the single e.m.f. e_{ab}, and obtain the same result as before. This is pointed out in order to emphasize the fact that, in applying the flux-cutting rule for induced e.m.f.'s, it is the *relative* velocity of the different parts of the coil, perpendicular to the magnetic field, which is of importance.

7. A hollow ebonite cylinder encloses a cylindrical bar-magnet, which is fixed. The cylinder rotates and in so doing winds a coil of wire about itself. The circuit of this coil is completed through a voltmeter. Would you expect any e.m.f. to be induced in the coil thus formed on the cylinder?

(*Note*. See footnote on p. 89.)

8. A circuit consists of two metal strips A and B (Fig. 44) connected to a sensitive voltmeter or galvanometer, with their other ends in contact as shown (the shaded portion represents an insulating block). A magnetized iron ring, of circular section, M, is pushed between the tips of the strips until it links with the circuit. During this motion the

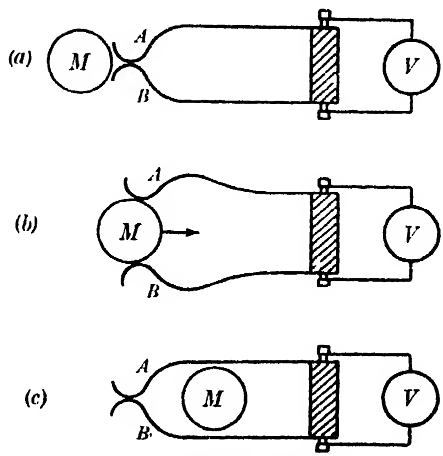


Fig. 44. Hering's experiment

voltmeter circuit is completed through the iron of the magnet; in position (a) the flux of the magnet does not link the circuit, while in position (c) it does. Consequently there has been a change in the amount of flux linking the circuit. Would you expect this to result in an induced e.m.f.?*

(Apply the conditions A and B, pp. 84 and 85. See also Ex. 21.)

9. A cylindrical bar-magnet (Fig. 45) rotates about its axis and a voltmeter is connected to two contacts, one sliding on the surface of the magnet at its centre, the other sliding on the shaft at the north-pole end as shown. The rotational axis is from north to south pole.

Would you expect an e.m.f. to be induced in the voltmeter circuit, and if so, in what direction will it act?

Ans. Yes. The direction is from the north-pole contact, to the central contact through the voltmeter. See Appendix III.

* This experiment is due to C. Hering. See *Tr. A.I.E.E.* XXVII (1908), Part II, p. 1341, or the *Electrician*, LXXV (July 16, 1915), p. 559.

10. If the coil of Example 5 carries a steady current of 10 amps and is fixed in the position where $\theta = 45°$, calculate the torque on the coil.

Solution. From 2(3) the torque $T = I\dfrac{d\phi}{d\theta}$ newton-metres.

Now
$$\phi = d^2 B \cos\theta,$$
$$\frac{d\phi}{d\theta} = -d^2 B \sin\theta,$$
so that
$$T = -Id^2 B \sin\theta.$$
Numerically,
$$T = 10 \times 10^{-2} \times 10^{-1} \times 0{\cdot}707 = 7{\cdot}07 \times 10^{-3} \text{ newton-metre}$$
$$= 7{\cdot}07 \times 10^{-3} \times 10^{7} = 7{\cdot}07 \times 10^{4} \text{ dyne-cm.}$$
$$= \frac{7{\cdot}07 \times 10^{4} \times 0{\cdot}0022}{981 \times 2{\cdot}54 \times 12} = 0{\cdot}0052 \text{ lb.-ft.}$$

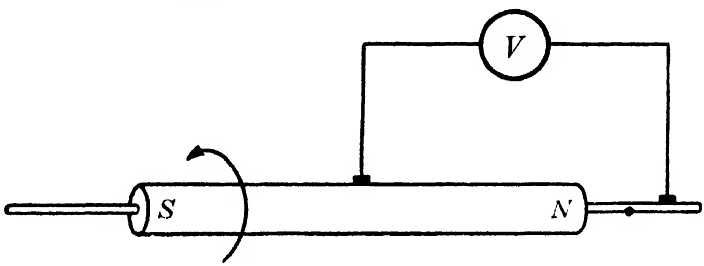


Fig. 45. Induction by rotating magnet

11. An electron of mass m and charge q_e moves unimpeded with a velocity v in, and perpendicular to, a magnetic field of density B. Show that it will move in a circular path of radius

$$R = \frac{mv}{Bq_e}.$$

12. An electron is projected with velocity v through a region in which there is a uniform magnetic field (B Wb/m^2) perpendicular to a uniform electric field (E volts per metre). The initial direction of the electron's motion is perpendicular to both fields, and it is found that it continues to travel in this direction without deviation.

Show that the velocity of the electron must be equal to E/B metres per sec.

13. A coil of 1000 turns links an alternating flux of maximum value 0·01 weber and frequency 50 cycles per sec. If the variation of the flux is sinusoidal, find the maximum value of the induced e.m.f.

Ans. 3142 volts.

14. A rectangular coil of 10 turns rotates between the two poles of an electro-magnet at 1200 r.p.m. If the flux ϕ passing across the air-gap is 10^{-2} webers (10^6 maxwells), and each active conductor of the

coil cuts the whole of this flux, find the mean value of the e.m.f. induced in the coil.

(Use the "rate of flux-cutting" rule of Section 8.)

Ans. 8 volts.

15. A heavy copper ring of rectangular section, of inner radius R_1 and outer radius R_2, is situated in a uniform alternating magnetic field normal to the plane of the ring. If the flux-density at any instant is

$$B = B_m \sin \omega t,$$

show that the mean value of the induced e.m.f. in the ring is the same as that induced in a circular ring of negligible section and radius given by

$$R = \sqrt{\frac{R_1^2 + R_1R_2 + R_2^2}{3}}$$

and situated in the same magnetic field.

16. A circular coil of one turn is situated in a uniform magnetic field of flux-density B, the plane of the coil being parallel to the direction of the field. Show that, if R is the radius of the coil and I is the current it carries, it experiences a torque given by

$$T = \pi R^2 I B.$$

17. A generator has a resistance of 1 ohm and generates an e.m.f. of 1100 volts. A load resistance of 10 ohms is connected across the terminals. Calculate the current, the total electrical power generated, the power delivered to the load, and the terminal p.d.

Ans. 100 amperes, 110 kilowatts, 100 kilowatts, 1000 volts.

18. A motor has a resistance of 0·1 ohm and is connected to a power supply at a p.d. of 200 volts. Find (a) the stand-still current, and (b) the back e.m.f. and the mechanical power delivered when taking a current of 100 amps.

Ans. 2000 amperes 190 volts, 19 kilowatts (25·5 h.p.).

19. A motor of resistance 0·1 ohm is connected to a battery of e.m.f. 200 volts, and internal resistance 0·05 ohm. The motor generates a back e.m.f. of 185 volts. Find the current it takes, the mechanical power delivered, and the terminal p.d.

Ans. 100 amperes, 18·5 kilowatts, 195 volts.

20. A short cylindrical metal tube surrounds a coaxial straight conductor, which carries a steady current Fixed brushes, making contact with the inner and outer surfaces of the tube, are connected to a voltmeter, and the tube is then moved along the axis of the wire. What difference do you expect, in the generated e.m.f. in the voltmeter circuit, if the tube is made of (a) copper, and (b) soft iron?

Solution. The boundaries of the walls of the tube are concentric circles, having the conductor at their centre. Thus the magnetic field provided by the iron tube is confined to the interior of the tube,

and can produce no effect in the stationary conductors. The field in the iron tube consists of two components: B_0 due to the current in the wire, and B_t due to the oriented magnetic axes of the iron atoms. In the case of the copper tube B_t is zero. Now when the tube moves, by our definition of moving magnetic fields it must be considered as taking its own field, B_t (due to its own atoms), with it, but as moving through the field B_0 of the current in the wire. Thus the generated e.m.f. should be proportional to B_0 alone, and as this is the same whether the tube is of iron or copper, the e.m.f. should be the same also.

Note. See also E. G. Cullwick, *Electromagnetism and Relativity*, pp. 134–5, and E. G. Cullwick, "An experiment on electromagnetic induction by linear motion", *Jl. I.E.E.* 85 (1939), p. 315.

21. A long hollow cylinder consists of an insulating material of dielectric constant K, its inner and outer cylindrical surfaces being faced with thin brass coatings. It is arranged to rotate in a constant magnetic field, produced by a stationary and co-axial solenoid surrounding the cylinder, parallel to the axis. An electrometer is connected by wires to sliding contacts on the inner and outer surfaces of the cylinder.

Show that the p.d. measured by the electrometer will be proportional to $1-1/K$.

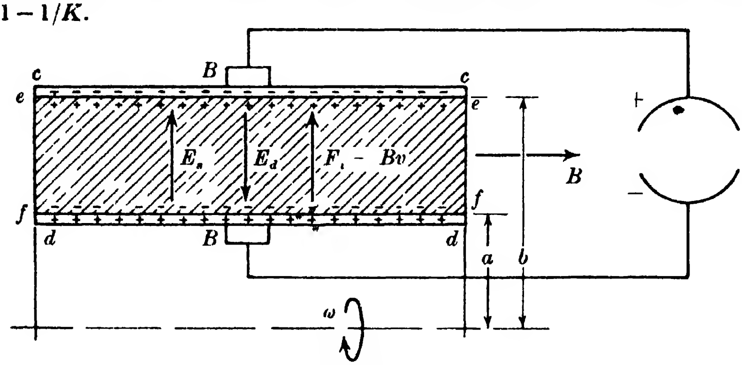


Fig 45a

Solution. In Fig. 45a the shaded area represents a section of the cylinder wall, the inner and outer radii being a and b, while $c-c$ and $d-d$ represent the metal coatings, whose thickness may be assumed to be negligible. Consider a point in the insulator distant r from the axis. If the angular velocity is ω this point will be moving with velocity $v=\omega r$ through the magnetic field B and hence the material of the insulator experiences an electromagnetic force, or motional intensity,

$$F_t = Bv. \tag{19}$$

The insulator is therefore polarized, the outer surface $e-e$ becoming positively charged and the inner surface $f-f$ negatively charged. These induced surface charges will tend to induce free charges on the inner surfaces of the metal coatings, which will become charged, provided

that charges can flow to them from the outer surfaces or from connected conductors, such as the electrometer plates and leads.

Under conditions of equilibrium there will be *three* components of electric force acting on the material of the insulator:

(1) the electromagnetic motional force $F_t = B\omega r$,

(2) the electrostatic field E_s of the charges on the brass coatings,

(3) the electrostatic field E_d of the charges displaced in the insulator by its polarization

The *polarizing force* acting on the insulator is due to the external sources, and is

$$F_0 = F_t + E_s = B\omega r + \frac{q}{2\pi\epsilon_0 r}, \tag{20}$$

where q is the charge per unit axial length of the coatings (see p. 38). The field of the displaced charges in the insulator is then, by 1(10a),

$$E_d = \left(B\omega r + \frac{q}{2\pi\epsilon_0 r}\right)\left(1 - \frac{1}{K}\right), \quad \text{opposed to } F_0. \tag{21}$$

The resultant electrostatic field between the stationary brushes $B - B$ is therefore

$$E = F_d - E_s = \left(B\omega r + \frac{q}{2\pi\epsilon_0 r}\right)\left(1 - \frac{1}{K}\right) - \frac{q}{2\pi\epsilon_0 r}$$

$$= B\omega r\left(1 - \frac{1}{K}\right) - \frac{q}{2\pi K\epsilon_0 r} \tag{22}$$

The potential difference between the brushes is then

$$V = \int_a^b E\,dr = B\omega\left(1 - \frac{1}{K}\right)\frac{b^2 - a^2}{2} - \frac{q}{2\pi K\epsilon_0}\log_e\frac{b}{a},$$

or

$$V = \phi n\left(1 - \frac{1}{K}\right) - \frac{Q}{C}, \tag{23}$$

where ϕ is the total flux passing through a cross-section of the cylinder, n is the speed in rev/sec , Q is the total charge on each metal coating, and C is the capacitance of the capacitor formed by the two coatings and the intervening dielectric (see p 38)

If C' is the capacitance of the electrometer, and if the stray capacitance of the outer surface of the coatings and of the leads is negligible, we have $Q = C'V$, since V is also the p d across the electrometer Whence we find

$$V = \phi n\left(1 - \frac{1}{K}\right)\frac{C}{C + C'} \tag{24}$$

This result has been experimentally verified by H A Wilson (*Phil Trans. Roy Soc* A, CCIV (1905), p 121)

A problem of great fundamental interest arises if the insulating cylinder in this problem is ferromagnetic. If the insulator has a magnetic permeability μ in addition to a dielectric constant K, and if the cylinder is sufficiently long for the simple relation $B \simeq \mu B_0$ to be used, where B is the total flux-density in the magnetic dielectric and B_0 that due to the magnetizing solenoid, then it may be shown that the relativistic theory of moving magnets leads to a result similar to 2(24) but with the factor $(1 - 1/K)$ replaced by $(1 - 1/\mu K)$, and this has been confirmed experimentally by H A and M. Wilson (*Proc. Roy. Soc.* A, 89 (1913),

pp. 99–106). By pre-relativistic Maxwell-Lorentz theory, however, equation 2(24) should still apply without any modification. (see H. A. Lorentz, *Lectures on Theoretical Physics*, III, pp. 304–5). A full theoretical treatment and discussion of the problem will be found in the author's *Electromagnetism and Relativity* (Longmans, Green and Co., Chapter 11), where it is shown that the relativistic theory may be interpreted very simply by the assumption that the component of the magnetic flux provided by the magnetic cylinder, $\phi_0(\mu-1)$, where ϕ_0 is the flux of B_0 through the dielectric, rotates with the cylinder and induces an e.m.f. of magnitude $n\phi_0(\mu-1)$ in the leads from the brushes to the electrometer, while in the cylinder itself, rotating in the component of flux-density B_0, an e.m.f. is induced of value $n\phi_0$. If we first neglect the effect of C', the capacitance of the electrometer, it follows that the e.m.f. in the cylinder produces a p.d. between the brushes of magnitude $n\phi_0(1-1/K)$, and if this is added to the p.d. $n\phi_0(\mu-1)$ produced in the leads we obtain a total p.d. equal to $n\phi_0(\mu-1/K)$ or $n\phi(1-1/\mu K)$. This neglects the effect of any charge Q on the metal coatings which arises from the connection of the electrometer and reduces the p.d. as in 2(23) above.

Applying these conclusions to Fig. 45 and Ex. 9 above, it follows that in the case of a long rotating cylindrical permanent magnet, in which the external component of field B_0 is absent, the e.m.f. is induced entirely in the leads.

22. An electron is accelerated along a circular orbit by an increasing magnetic flux which treads the orbit centrally. Show that the electron will continue to move in a path of constant radius, if the flux-density *at* the path is always one-half the mean flux-density enclosed *by* the path.

Solution. This is the principle of the Betatron. Let R be the radius of the orbit, B the mean flux-density over the orbit and ϕ the flux linking the orbit. Then an e.m.f. is induced around the orbit of magnitude

$$e=-\frac{d\phi}{dt}=-\pi R^2\frac{dB}{dt}, \tag{25}$$

and since the magnetic field is taken to be symmetrical about the axis this means that an electric field E is induced which is concentric with the axis and of the same magnitude at all points of the orbit. The e.m.f. is therefore given by $e=2\pi RE$, so that

$$E=\frac{R}{2}\frac{dB}{dt}. \tag{26}$$

If B' is the magnetic flux-density at the orbital path and v the velocity of the electron, the latter experiences a radial force qvB', where q is the electronic charge. If m is the electronic mass we therefore have

$$qvB'=\frac{mv^2}{R}, \quad \text{and} \quad B'=\frac{mv}{qR}. \tag{27}$$

The force accelerating the electron is qE and this is equal to the rate of change of momentum. Hence

$$qE=\frac{qR}{2}\frac{dB}{dt}=\frac{d}{dt}(mv). \tag{28}$$

We assume that the electron starts from rest when $B=0$, and integration of 2(28) then gives

$$mv=\frac{qRB}{2} \tag{29}$$

so that, from 2(29) and 2(27),

$$B=\frac{2mv}{qR}=2B', \tag{30}$$

which is the required relation between B and B'.

It should be noted that we have not assumed the electronic mass, m, to be a constant, for 2(28) is valid even if m is a function of velocity. Equation 2(30) therefore still applies if the electron is accelerated to velocities comparable with c, the velocity of light, in which case its mass as used in 2(28) is given by $m=m_0(1-v^2/c^2)^{-\frac{1}{2}}$. If, however, we use the relation *force = mass × acceleration*, the law for the variation of electronic mass with velocity is as given on p. 64.

CHAPTER III

THE MAGNETIC FIELD OF THE ELECTRIC CURRENT

1. The magnitude of the magnetic field of a moving charge and a current element.

So far we have merely accepted the magnetic field as a vector quantity at a point which is measurable by an observer, relative to whom electric charges are moving. We have now to correlate the magnitude of a given field with that of the current to which it is due.

It is of importance, first, to reiterate the fact that electric and magnetic fields, as defined in the previous pages, are not necessarily *absolute* quantities at a point, since they are defined in terms of effects which are apparent to an observer to whom all velocities must be referred. Now all velocities are relative imagine two observers, A and B, one stationary on the ground and the other travelling in a train at a uniform speed. Suppose that each observer has identical apparatus for measuring magnetic fields, and that they each take measurements of the magnetic field of the same moving charge. The velocity of the charge will not be the same for each observer (that is, the charge will represent a different *current element*, $I\,\delta l$, for each observer), so that the value assigned to the magnetic field by A will not agree with that obtained by B. Hence when we speak of the magnetic field of a charge q moving with velocity v, we mean the *magnetic field which is apparent to the observer relative to whom the charge moves with velocity v.*

A further point which requires notice is the conception of "moving" fields. We have spoken of a conductor moving across a magnetic field, and have taken the precaution to stipulate that this motion is that of the conductor *relative to the system of currents to which the field is due*. But suppose that we are situated in a uniform magnetic field and have no means of observing the state of motion of the parental currents, can a prescribed "motion" of the field have any meaning for us?

According to equation 2(12 b), if there is relative motion between us and the field, we should experience an electric field perpendicular both to the lines of force of the moving field and to the direction of motion, but there may be neighbouring electric charges whose presence we cannot detect, for their electric field may be indistinguishable from that due to the moving magnetic field. Hence if we cannot measure or calculate the velocity of the conductor relative to the system of currents which causes the magnetic field, we cannot measure or calculate the velocity of the field itself, and a velocity which cannot be measured or calculated has no physical meaning for us.

The same limitation applies to the electro-static field: we may speak of the velocity of such a field only in terms of the velocity of the charges from which the field emanates.

Let us now correlate certain facts which we have already studied.

(*a*) *An observer moving across a magnetic field observes an electric field.* The three vectors of velocity (v), magnetic field (B), and electric field (E), are related as in Fig. 46, (v, B, E) forming a right-handed set.

Now let the velocity v be measured by the second observer. That is, it is the velocity of the magnetic field relative to the observer who measures E. The relation between the vectors is now shown in Fig. 47, which is derived from Fig. 46 by reversing the vector v.

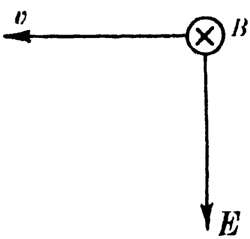


Fig. 46

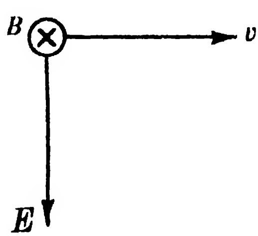


Fig. 47

Fig. 46. (B is vertically downwards, due to a stationary current circuit. E is measured by an observer whose velocity relative to the circuit is v.)

Fig. 47. (E and B are measured by a stationary observer, relative to whom the magnetic field (and the circuit to which it is due) is moving with velocity v. The three vectors (B, v, E) form a right-handed set.)

The electric field observed in either case is

$$E = Bv \sin \alpha, \qquad 2(11a)$$

where α is the angle between the vectors B and v.

(*b*) *A charge, moving relatively to an observer, causes that observer to experience a magnetic field.* The lines of force of the magnetic field are circles concentric with the direction of motion of the charge (Fig. 48).

Consider the instant when the observer is at the point P, where the line joining P to the charge is perpendicular to the direction of motion. The electric field E at P due to q is radially outwards from the charge and also perpendicular to the direction of motion, and we may think of this field as moving with the charge from left to right.

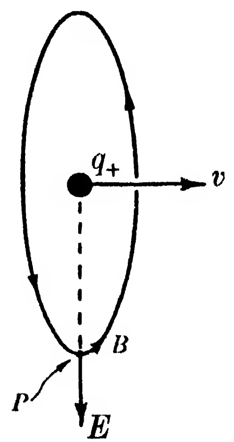


Fig 48

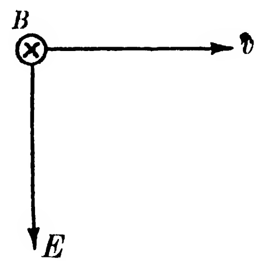


Fig 49

Fig. 48 (The positive charge q is moving with velocity v relatively to an observer at P. The observer measures a magnetic field which is normal to the paper and downwards.)

Fig. 49 (B and E are measured by a stationary observer, relative to whom the electric field E is moving with velocity v.)

The magnetic field at P may then be considered as due to the motion of the electric field E, moving past P with velocity v. The directions of the three vectors E, v and B are shown in Fig. 49.

Now Figs. 47 and 49 are identical, and one case can be transformed into the other by interchanging the letters B

and E in the explanatory notes below them. For the magnetic field due to a moving electric field, then, we are tempted to suggest a quantitative relation similar in form to that for the complementary case. That is, similar to equation 2(11 a) with letters B and E reversed. We therefore write:

$$B_0 = kEv \sin \alpha, \tag{1}$$

where k is some constant, and α is the angle between E and v.

We add the suffix 0 to B in this equation to denote that B_0 is the value of the flux-density produced in a "non-magnetic" medium (free space). When the medium consists of a material substance, the moving charge will in general affect the atoms of the substance. There may be some reorientation of the magnetic axes of the atoms, and some change in the atomic magnetic moments themselves, so that the *average* value of the flux-density in the inter- and sub-atomic space is a function of the properties of the material, and is called the *induction*, B. We allow for this exceedingly complex effect by means of a statistical factor μ, the *relative permeability* of the material, given by

$$\mu = \frac{B}{B_0}, \tag{2}$$

where B and B_0 are to be found in a case where the *configuration* of the magnetic field is unaltered by the introduction of the material substance. As we shall see in Chapter IV, one practical case where this condition is fulfilled is that of the uniformly wound toroidal ring. Under these conditions we may speak of B_0 as the *magnetizing flux-density*.

Now consider a point P (Fig. 50) distant r from the moving charge, such that the radius vector r makes an angle α with v.

Then, if $v \ll c$,
$$E_p = \frac{q}{4\pi K \epsilon_0 r^2}, \tag{1(10)}$$

so that if 3(1) is true we must have

$$B_0 = \mu_0 \frac{qv}{4\pi r^2} \sin \alpha, \tag{3}$$

where $\mu_0 = k/K\epsilon_0$, a quantity whose significance will appear later, and provided $v \ll c$,

In terms of this new factor, μ_0, the value of B_0 for a current element $I\,\delta l$ is therefore

$$B_0 = \mu_0 \frac{I\,\delta l}{4\pi r^2} \sin\alpha. \qquad (4)$$

The value of μ_0. Now the magnetic field of a current is found to be independent of the dielectric constant, K, of the surrounding medium, so that the factor μ_0 is independent of K. It is, indeed, a quantity of fundamental importance in electromagnetic theory, and is usually called the *permeability of free space*. The reason for this name lies in the viewpoint of the old material-medium hypothesis, and a better name is *primary magnetic constant*.

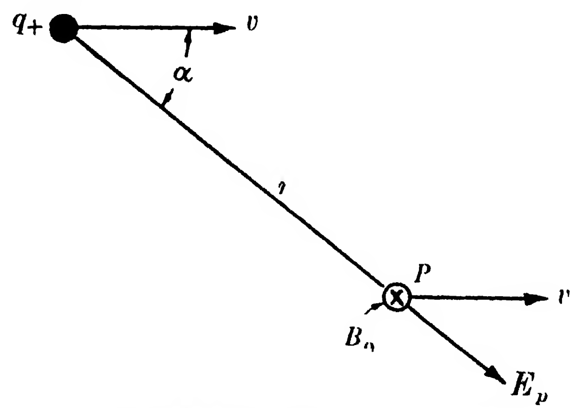


Fig. 50. Magnetic field of moving charge

Its value is found, by experiment, to be such that

$$\epsilon_0 \mu_0 = \frac{1}{c^2},^* \qquad (5)$$

where c is the velocity of light. Thus

$$\mu_0 = \frac{1}{\epsilon_0 c^2}, \qquad (5a)$$

and experiment also shows that the form of the equations 3(3) and 3(4), above, is correct when applied to closed circuits.†

* This relation is deduced, by means of the restricted theory of relativity, in Section 7 below.

† Equation 3(3) is correct only if the velocity v is very much less than the velocity of light.

In the rationalized m.k.s. system the value of μ_0 is $4\pi \times 10^{-7}$, and will be seen later to have the dimensions *henrys/metre*, the *henry* being the m.k.s. unit of inductance. The value of ϵ_0 then follows from 3(5) and is $8{\cdot}854 \times 10^{-12}$ farad/metre, since $c = 2{\cdot}998 \times 10^8$ m/sec. In the classical unrationalized c.g.s. electromagnetic system the value of μ_0 is unity.

2. The effects of moving fields.

Bearing in mind our definition of moving fields, we now have, as the result of the motion of current-circuits, magnets and electrically charged bodies:

(A) *At a point in free space, or in a non-magnetic insulator for which K is unity.*

(1) Due to a magneto-static field, of value B and velocity v, relative to the point:

There is an electric field given by

$$E = Bv \sin \alpha, \quad \text{or} \quad \mathbf{E} = \mathbf{B} \times \mathbf{v} \qquad 2(11a)$$

at the point.

(2) Due to an electro-static field, of value E and velocity v, relative to the point:

In 3(1) above we put $k = \epsilon_0 \mu_0 = 1/c^2$, so that there is a magnetic field

$$B = \frac{Ev}{c^2} \sin \alpha, \quad \mathbf{B} = \frac{\mathbf{v} \times \mathbf{E}}{c^2} \text{ at the point.} \qquad (6)^*$$

If B (in case 1) or E (in case 2) is due to various sources with different velocities, the equations must be applied to each component, with its appropriate velocity, and the results combined vectorially.

(B) *At a point fixed in a material body of relative permeability μ and dielectric constant K.*

(1) *Due to a moving magneto-static field,* B_0. Let B be the resultant flux-density at the point. Then

$$B = B_0 + B_i, \text{ vectorially (see p. 194)}$$

* Thus an iron needle moving through an electro-static field should become magnetized. So clear-sighted was Faraday that on March 26, and again on April 6 and 7, 1832 he tried such an experiment. It failed, but we see from this equation that the effect would be extremely small. See *Faraday's Diary*, Vol. I, pp. 425–6.

where B_0 is the component due to sources outside the body, and B_i is the component due to oriented magnetic axes of atoms in the body.

The component B_i is stationary relative to the point considered, so that 2(11a) must be applied to the component B_0 only, where if necessary the various sources of B_0 must be treated separately.

(In the special case where the boundaries of the body are parallel everywhere to the lines of force of the magnetic field, we have
$$B_0 = \frac{B}{\mu}, \quad \text{and} \quad B_i = B\left(1 - \frac{1}{\mu}\right).)$$

Since the material has a dielectric constant K, equation 2(11a) will give
$$E_0 = B_0 v \sin \alpha$$

and the resultant electric field at the point will be
$$E = E_0 + E_d, \text{ vectorially.} \quad \text{(p. 20)}$$
where E_0 is the component due to the moving field B_0, and does not exist for an observer at rest relative to the source of B_0, and E_d is the electro-static field of the charges displaced by E_0. This is the same for all observers provided that their velocities relative to the point are very small compared with c.

(In the special case where $B_0 = B/\mu$, E_0 is perpendicular to the surfaces of the body, so that
$$\mathbf{E} = \frac{\mathbf{E}_0}{K} = \frac{\mathbf{B}_0 \times \mathbf{v}}{K}. \qquad (6a).)$$

If the material is a conductor:

$E = 0$ if no current can flow

or $\quad E = \rho J$ (1(35)) if a current of density J is flowing.

(2) *Due to a moving electro-static field*, E_0. Let E be the resultant electro-static field at the point.

Then
$$E = E_0 + E_d, \text{ vectorially,}$$
where E_0 is the component due to sources outside the body, and E_d is the component due to displaced charges in the body.

The component E_d is stationary relative to the point considered, so that 3(6) must be applied to the component E_0

only, where if necessary the various sources of E_0 must be treated separately.

Since the material has a relative permeability μ, equation 3(6) will give

$$B_0 = \frac{E_0 v}{c^2} \sin \alpha$$

and the resultant magnetic field at the point will be

$$B = B_0 + B_i, \text{ vectorially,}$$

where B_0, the component due to the moving field E_0, does not exist for an observer at rest relative to the source of E_0, but the component B_i, due to oriented magnetic axes of atoms in the body, is the same for all observers provided that their velocities relative to the point are very small compared with c.

(In the special case where E_0 is perpendicular to the surfaces of the body, B_0 will be parallel to the surfaces, so that

$$\mathbf{B} = \mu \mathbf{B}_0 = \frac{\mu(\mathbf{v} \times \mathbf{E}_0)}{c^2} \qquad (6b).)$$

Notice that, since $B = \mu_0 H$ in free space (Section 17 below), and $D = \kappa_0 E$, equation 3(6) can be written

$$H = Dv \sin \alpha, \quad \text{or} \quad \mathbf{H} = \mathbf{v} \times \mathbf{D}, \qquad (7)$$

which is mathematically symmetrical with 2(12b).

Notice also that 3(6) and 3(3) give *alternative* methods for calculating the magnetic field due to moving charges. Another alternative method of obtaining the magnetic field of moving charges, in the case where the velocities of the charges are unknown, but where their motion results in a changing electrostatic field, is by means of the concept of displacement current (see Section 14 (d) below).

For application of the above analysis, see Chapter II, Section 10, and Ex. 20; also Chapter III, Ex. 3.

3. The magnitudes of the electric current, and its magnetic field, in a metallic conductor, are independent of the relative velocity between conductor and observer.

It has been pointed out that the magnetic field of a moving charge is proportional to the *relative* velocity of charge and

observer, so that for an observer travelling with the charge no magnetic field exists. This does not apply, however, to currents flowing in metallic conductors.

Consider a current I flowing in a metallic conductor (Fig. 51). Then we regard this as due to a number of electrons (of total charge q_- per unit length of the conductor) moving with a mean velocity v, relative to the conductor, in a direction opposite to the conventional direction of the current I.

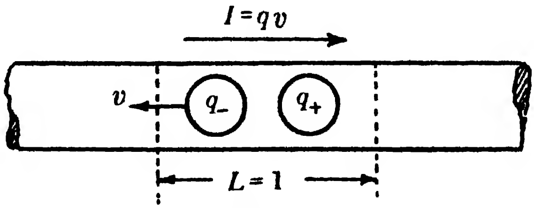


Fig. 51

Now if we assume that there is no aggregate charge in any finite volume of the conductor,* there must be an equal positive charge, q_+ per unit length, which is at rest relative to the conductor.

Suppose that the conductor is moving with velocity v_0 in any direction relative to the observer. Then the charges per unit length, moving relatively to the observer, are

$$q_- \text{ with velocity } v+v_0,$$

where v and v_0 are added vectorially, and

$$q_+ \text{ with velocity } v_0.$$

The current I_0 relative to the observer is therefore

$$I_0 = -q(v+v_0)+qv_0 = -qv = I.$$

Thus the current in the wire is independent of the motion v_0, and it follows that the magnitude of the magnetic field,

* Actually this is not quite true, for electro-static fields exist, in machines, between conductors carrying currents, showing that all regions cannot contain equal numbers of positive and negative charges (see Chapter II, Section 10). This net charge is usually so small, however, that it may be neglected for the purpose of this argument.

measured by an observer at any point, is independent of any steady motion of the circuit which carries the current.*

In the case of Fig. 47, the magnitude of the magnetic field will be the same whether measured by the stationary observer, or by an observer travelling with the current-circuit.* The stationary observer, however, is situated in an electric field of intensity $E = Bv\sin\alpha$, whereas for the observer travelling with the circuit no such field exists.

In electrical engineering extensive use is made of current circuits having relative motion. For instance, in generators and motors the armature current-circuits move relatively to the field current-circuits; in the theory of these machines the same value is taken for the armature current, and its magnetic field, whether these are referred to the moving conductors (e.g. in calculating the power lost in heating these conductors) or to the external stationary circuit (e.g. in calculating the output or input of the machine). From the above discussion we see that this is correct.

As a further example, we may take the interesting case of a rotating solenoid (Fig. 52). The solenoid is arranged to rotate about its axis, and current is led through by means of a stationary battery connected to contacts sliding on slip-rings at each end of the solenoid.

The current in the circuit is measured by an ammeter A_1 in the stationary part of the circuit, and a second ammeter A_2 can be imagined to be connected in the solenoid itself, and rotating with it.

From the principle discussed above, it follows that the current in the solenoid is independent of any motion it may

* Provided the velocity of the circuit, relative to the stationary observer, is considerably less than c, the velocity of light. Suppose that the field measured by an observer at rest relative to the circuit is B, then the relativity theory shows that the field observed by an observer travelling with velocity v relative to the circuit, and perpendicular to the field, will be equal to

$$B' = \frac{B}{\sqrt{1-v^2/c^2}}.$$

For the usual velocities found in electrical machines the difference is quite negligible.

possess relative to an observer (we may regard the ammeter as the observer) so we shall expect the two meters A_1 and A_2 to indicate the same values. Further, if we measure the magnetic field of the solenoid at any stationary point P, it follows that the result should be independent of any rotation of the coil.*

The electric field induced by the rotation of a solenoid. Certain experiments, such as those of Kennard and Pegram,† show that no detectable electric field is induced inside a rotating solenoid which carries a constant current. This is consistent

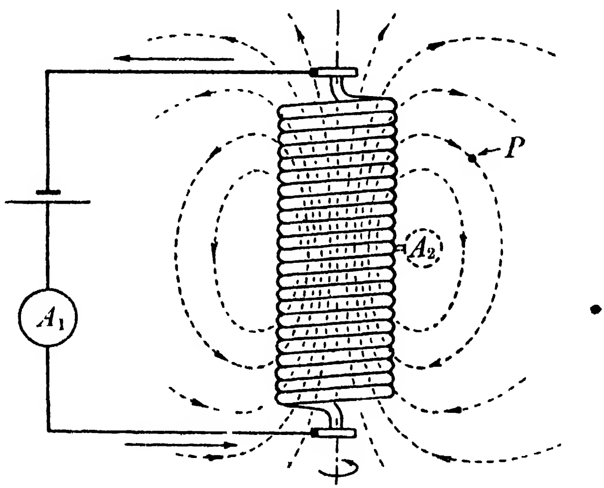


Fig. 52. Rotating solenoid

with accepted electro-magnetic theory, for the conditions necessary for the electro-magnetic induction of an electric field, as discussed in Chapter II, are not present in such experiments. The case is not comparable with that of a rotating cylindrical magnet since each element of the latter can be considered to be a complete magnetic field-source in linear motion, which is the condition for the motional induction of an electric field.

* As verified by A. Föppl (*Ann. d. Phys.* XXVII, 1886, p. 410), and by E. L. Nichols and W. S. Franklin (*Amer. Journ. Sci.* XXXVII, 1889, p. 103).

† *Phil. Mag.* XXXIII (1917), p. 179; *Phys. Rev.* X (1917), p. 591.

4. The magnetic field of the current in an infinitely long straight conductor. (In a medium of relative permeability $\mu = 1$.)*

Let the current in the wire be I. The field at any point P (Fig. 53), distant R from the conductor, is obtained by summing the contributions of all elementary lengths δl of the conductor, and we assume that the thickness of the wire is so small compared to the distance R that the whole of the moving charge in an indefinitely short element is at the same distance, r, from P.

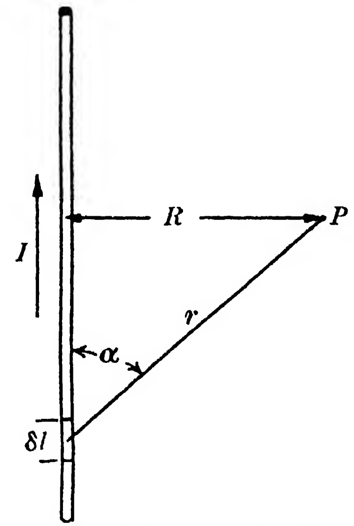


Fig. 53. Long straight conductor

Consider an element δl where the line joining the element to the point P makes an angle α with the direction of current flow. Then, from 3(4), the magnetic field at P due to this current element will be perpendicular to the paper and given by

$$\delta B_p = \mu_0 \frac{I\,\delta l}{4\pi r^2} \sin\alpha$$

$$= \mu_0 \frac{I\,\delta l}{4\pi R^2} \sin^3\alpha.$$

* In this and the following sections the medium is taken to be non-magnetic. If the circuit considered in any case is surrounded by a homogeneous magnetic medium, the equations give B_0 and not B.

Now in Fig. 54 the perpendicular CD dropped from C to AP, when δl is very small, is equal to

$$\delta l \sin\alpha = r\delta\alpha = \frac{R\delta\alpha}{\sin\alpha}$$

or $$\delta l \sin^2\alpha = R\delta\alpha,$$

neglecting second-order differentials.

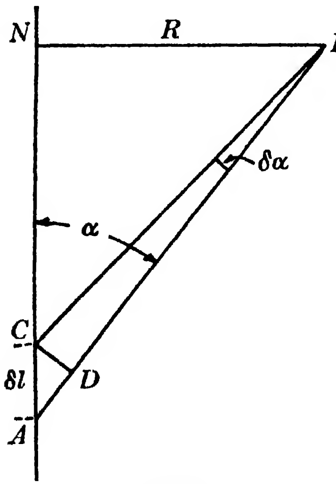


Fig. 54

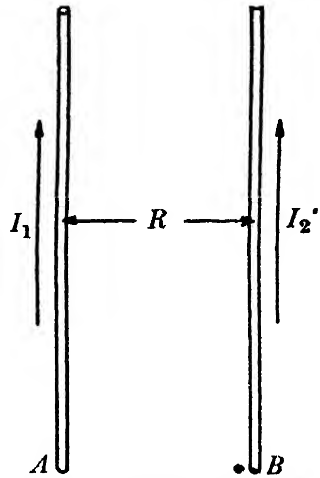


Fig. 55. Parallel conductors

Hence $$\delta B_p = \mu_0 \frac{I}{4\pi R} \sin\alpha\,\delta\alpha,$$

and the field at P due to the complete conductor is

$$B_p = \mu_0 \frac{I}{4\pi R}\int_0^\pi \sin\alpha\,d\alpha$$

$$= \mu_0 \frac{I}{2\pi R}. \qquad (8)^*$$

* We may think of the electric field at P, due to all the *moving* charges in the conductor, as moving with the same mean velocity. Hence B_p can be calculated from equation 3(6). Actually, of course, there is no net electric field at P due to the steady current I, since that due to the moving negative charges in an element is cancelled by that due to the equal positive charge which is stationary in the element of wire. In using 3(6) we take the field $-E$ of the negative charges to be moving, and the field $+E$ of the positive charges to be at rest. See Ex. 10 at the end of this chapter.

5. The force between two infinitely long parallel conductors, each carrying a steady current.

Let the conductors A and B (Fig. 55) be distant R apart, and let them carry steady currents I_1 and I_2 respectively. Let the medium be non-magnetic.

Then the current I_1 sets up a field at all points of the path of I_2 equal to

$$B = \mu_0 \frac{I_1}{2\pi R}.$$

so that (equation 2(9)) each unit length of B experiences a force, directed towards A, given by

$$F = \mu_0 \frac{I_1 I_2}{2\pi R}, \quad \text{per unit length of the conductor} \qquad (9)$$

and the conductor A experiences an equal force, per unit length, directed towards B. If the currents flow in opposite directions, the force F is one of repulsion.

It may be noted that equation 3(9) provides the basis for the internationally accepted definition of the ampere.

6. The force between two charges moving in parallel paths.

(A) *The magnetic force between the charges.* Let two positive charges q_1 and q_2 be situated, at a given instant, at the points P and Q (Fig 56a). Let v_1 and v_2 be their velocities, both assumed to be very small compared with c, relative to the observer, and r the distance between them.

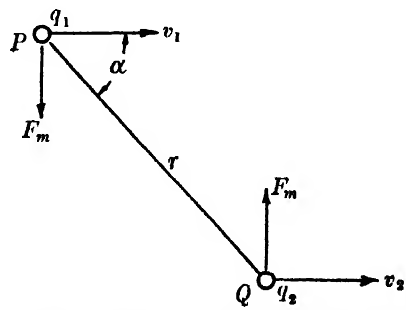


Fig. 56a. Charges moving in parallel paths

Then, by 3(3), the charge q_1 sets up a magnetic field at Q:

$$B_Q = \mu_0 \frac{q_1 v_1}{4\pi r^2} \sin\alpha,$$

which is perpendicular to the paper

The charge q_2 is moving at right angles to this field. Hence, by 2(10), it should experience a force

$$\begin{aligned} F_m &= B_Q q_2 v_2 \\ &= \mu_0 \frac{q_1 q_2 v_1 v_2}{4\pi r^2} \sin\alpha. \qquad (10) \end{aligned}$$

and the charge q_1 experiences an equal force due to its motion in the magnetic field of q_2. The forces F_m are in the plane of the paths, and perpendicular to the direction of motion. This law has not been verified for individual charges.

(B) *The total force between two charges, moving side by side in parallel paths with equal velocities.* Now let $\alpha = \pi/2$ and $v_1 = v_2$. Then the magnetic force between the charges becomes a simple attraction given by

$$F_m = \mu_0 \frac{q_1 q_2 v^2}{4\pi r}.$$

so that if we assume the electro static repulsion, F_s, to be unaltered by the motion, we would calculate the net repulsion between the charges to be

$$\begin{aligned} F = F_s - F_m &= \frac{q_1 q_2}{4\pi\epsilon_0 r^2} - \mu_0 \frac{q_1 q_2 v^2}{4\pi r^2} \\ &= F_s\left(1 - \frac{v^2}{c^2}\right), \quad \text{since } \epsilon_0\mu_0 = \frac{1}{c^2} \end{aligned}$$

Since we have assumed that $v \ll c$, it would appear that the modification due to the motion is quite negligible, but there are two reasons why we cannot be satisfied with this simple conclusion. The first is that, in high-voltage cathode-ray tubes, we have to deal with electrons which move with speeds comparable with the speed of light, and the second is that, by using the more exact theory (which involves Einstein's restricted theory of relativity) we are able to gain an insight into the nature of the magnetic forces between current-carrying circuits.

It is impossible, in a book of this kind, to give a full account of the restricted relativity theory, and it will be sufficient to state the modification it imposes upon our elementary equations for the simple case which we are considering. This is

> Suppose an observer to be at rest in an electro-static (or magnetic) field which he observes to have a value E (or B). Then another observer, moving with uniform velocity v *perpendicular* to the field, will find that it has the value $E/\sqrt{1-\beta^2}$ (or $B/\sqrt{1-\beta^2}$), where $\beta=v/c$, but if his motion is *parallel* to the field, then its value will appear to be unaltered.
>
> If, however, a charge q is moving through the field with velocity v relative to the first observer, the physical phenomena as measured by that observer will be consistent with the assumption that the force experienced by the charge is still given by Eq (or by Bqv).

Now apply these principles to our simple case of two charges moving side by side. The charge q_1 will be credited by an observer, stationary at the point where q_2 is momentarily situated, with an electric field

$$E'=\frac{q_1}{4\pi\epsilon_0 r^2}\frac{1}{\sqrt{1-\beta^2}},$$

and a magnetic field

$$B'=\frac{\mu_0 q_1 v}{4\pi r^2}\frac{1}{\sqrt{1-\beta^2}}.$$

The charge q_2, moving in these fields, experiences forces.

(a) an electro-static repulsion

$$F'_s=\frac{q_1 q_2}{4\pi\epsilon_0 r^2}\frac{1}{\sqrt{1-\beta^2}},$$

(b) a magnetic attraction

$$F'_m=\frac{\mu_0 q_1 q_2 v^2}{4\pi r^2}\frac{1}{\sqrt{1-\beta^2}}.$$

There is a net repulsion:

$$F'=F'_s-F'_m=\frac{q_1 q_2}{4\pi\epsilon_0 r^2}\left(\frac{1}{\sqrt{1-\beta^2}}-\frac{v^2}{c^2}\frac{1}{\sqrt{1-\beta^2}}\right)$$

$$=F_s\sqrt{1-\beta^2}, \qquad (10a)$$

where F_s is the electro-static repulsion measured by an observer moving with the charges.

Equation 3(10*a*) is, in fact, a general relativistic result for the relation between the force acting on a stationary body and the same force as measured by an observer moving with velocity v relative to the body and at right angles to the force.

7. The magnetic force between current-carrying conductors calculated from the relativity modification of electro-static forces; a theoretical development of the relation $\epsilon_0\mu_0 = 1/c^2$.

In explaining the forces between conductors carrying electric currents, it is usually assumed that these are due to the interaction of the current in one conductor with the magnetic field due to the other, and vice versa, and that electro-static forces are not present. It is interesting to see whether the phenomena can be calculated from the relativity modification of electro static forces, with no reference to magnetic fields.

Consider a long straight conductor, carrying a steady current I_1, and an element $I_2\,\delta l_2$ of a parallel conductor carrying a current I_2. We wish to calculate the force acting on this current-element. The electric charges in the first conductor may be split up into pairs, as shown at A, B in Fig. 56*b*, of which the

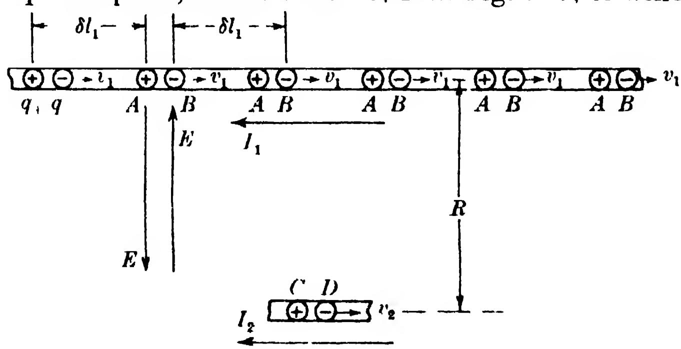


Fig 56*b*. Parallel currents.

positive charges A are fixed relative to the conductor while the negative charges B move with velocity v_1 along the conductor in the direction opposite to that of the conventional current.

We assume that all these charges have the same magnitude q so that the conductor as a whole carries no aggregate charge. This means that if δl_1 is the spacing between adjacent stationary positive charges, the spacing between adjacent moving negative charges must also be δl_1. The element of the second conductor is represented by the pair of charges C and D, of which C is positive and stationary and D is negative and moving with velocity v_2 in the same direction as v_1. The current I_2 is opposed to v_2 and the moving charge D represents a current element $I_2\,\delta l_2 = qv_2$.

To an observer for whom the two conductors are stationary, the current I_1 causes no resultant electric field and exerts no force upon the stationary charge C in the current-element C, D. We may confirm this by noting that the line of positive charges A constitutes a uniform linear distribution of charge of amount

$$\rho = q/\delta l_1 = I_1/v_1 \quad \text{per unit length,}$$

and that they produce a component of electric field, radially outwards as shown, which from the Theorem of Gauss has an intensity

$$E = \frac{\rho}{2\pi\epsilon_0 R} = \frac{I_1}{2\pi\epsilon_0 R v_1} \tag{10b}$$

at points distant R from the axis of the conductor. The moving negative charges have a linear density $-\rho$ and therefore cause an exactly equal but opposing field, radially inwards, and the resultant electric field is zero.

We are then left with the problem of calculating the force on the moving negative charge D without using the concept of the magnetic field. We must therefore refer the situation to the reference system in which the charge D is stationary, i.e. to a reference system S' which is moving with velocity v_2 relative to the first reference system, S, in which the conductors are at rest. In order to do this, we require the relativistic relations for the electric field given on p. 135, the relativistic law of transverse force given by 3(10a), and also the law for the relativistic transformation of velocity. We need the latter in order to find the velocity of the charges B as observed in the reference system S', in which the charge D is stationary. By

ordinary Newtonian kinematics this is merely $v_1 - v_2$, but the Restricted Theory of Relativity requires it to be

$$w = \frac{v_1 - v_2}{1 - v_1 v_2/c^2} \qquad (10c)^*$$

The force experienced by the charge D, as observed in the reference system S', will be proportional to the resultant electric field at D due to the positive and negative charges, A and B, in the long conductor which is moving with velocity $-v_2$ in S'. This resultant field is not zero.

First consider the component of field, as observed in S', due to the positive charges A. Since these are now moving with velocity v_2 it follows from the rules given on p. 135 that they cause, in S', a field, radially outwards from the conductor, of intensity

$$E_1 = \frac{E}{\sqrt{1 - v_2^2/c^2}}, \qquad (10d)$$

where E is given by 3(10b). To find the field of the negative charges B, as observed in S', we must first find this field as observed in the reference system in which the charges B are stationary. Remembering that this field is E as observed in the system S in which they are moving with velocity v_1, it follows that in the system in which they are stationary they have a weaker field, $E\sqrt{1 - v_1^2/c^2}$ In the system S', in which D is stationary, this field is moving with velocity w as given by 3(10c), and hence has the value, radially inwards,

$$E_2 = E\frac{\sqrt{1 - v_1^2/c^2}}{\sqrt{1 - w^2/c^2}}, \qquad (10e)$$

and this reduces, after a little algebra, to

$$E_2 = \frac{E(1 - v_1 v_2/c^2)}{\sqrt{1 - v_2^2/c^2}}. \qquad (10f)$$

It follows that the current I_1 causes, in the system S', a

* See E. G. Cullwick. *Electromagnetism and Relativity* (Longmans, Green and Co.) p. 76.

resultant electric field, radially outwards,

$$E' = E_1 - E_2 = \frac{E}{\sqrt{1 - v_2^2/c^2}} \frac{v_1 v_2}{c^2}, \qquad (10g)$$

and therefore the negative charge D experiences a force, in S', $\mathbf{F}' = -q\mathbf{E}'$, i.e towards the other conductor. In the original reference system S, however, in which both conductors are stationary, this force must be taken to be, as in 3(10a),

$$F = F' \sqrt{1 - v_2^2/c^2} = \frac{qEv_1 v_2}{c^2}. \qquad (10h)$$

This is the totàl force on the current-element C, D, and if we now substitute the value of E given by 3(10b) and also put $qv_2 = I_2 \delta l_2$, we obtain

$$F = \frac{I_1 I_2 \delta l_2}{2\pi R \epsilon_0 c^2}. \qquad (10i)$$

But from 3(8) and 2(9) this force is also given, in terms of the magnetic field, by

$$F = \frac{\mu_0 I_1 I_2 \delta l_2}{2\pi R}, \qquad (10j)$$

so by equating these two expressions for the force we obtain

$$\mu_0 = 1/\epsilon_0 c^2 \quad \text{and} \quad \epsilon_0 \mu_0 = 1/c^2 \qquad (10k)$$

It thus appears that the magnetic forces between conductors carrying electric currents may be explained by the extremely small modification of electro-static forces due to the small drift velocities of the moving electrons in the conductors.* It is true that we have neglected the unknown random velocities of the conduction electrons, but we can consider the moving charge B to be that in an elementary volume of the conductor, large enough to contain a very great quantity of conduction electrons, whose random velocities may then be assumed to cancel. That the resultant "magnetic" forces should be of practical magnitudes is due to the almost inconceivably large density of charge in metallic conductors. (See Chapter I, Part II, Section 2.)

We are now able to appreciate the enormous difference in magnitude between the electro-static and magnetic forces associated with a given electric charge. If it were possible to

* See also H. Pelzer and S Whitehead, *Br. Jl. Ap Phys.*, 2 (1951), p. 330

locate two isolated stationary charges, each of one coulomb, one metre apart, equation 1(2) tells us that the mutual force would be roughly a *million tons weight*, whereas if the same charges were moving in parallel conductors, one metre apart, with velocities of one metre per second, their contribution to the force of attraction between the conductors, due to this motion, would be only about 2×10^{-8} lb. wt., or $1/c^2$ of the electro-static force. This fact is, of course, inherent (but obscured) in the well-known ratio of the c.g.s. electro-magnetic and electro-static units of charge, this ratio being equal to c.

In the above, we have restricted our discussion to the case of parallel current elements, since it is only in this case that the forces experienced by two elements are equal and opposed in parallel paths. If, for instance, we apply the usual equations to the case of two elements, A and B, which are perpendicular to one another, such that the centre of B is on the axis of A, we are led to the conflicting conclusion that, while A is in the magnetic field of B and so should experience a force, the magnetic field due to A is zero at the point occupied by B, which therefore cannot experience a force. This seems to be a violation of Newton's third law of motion (that action and reaction are equal), but we must remember that we are dealing with an *element* of a circuit only. When we apply our theory to a complete circuit, acting upon another complete circuit, the result is consistent with Newton's third law. As a matter of fact, it is well known that quite different laws of force between current elements may be used, all of which give correct results for complete circuits.*

8. The magnetic field at the centre of a circular loop.

Let the radius of the loop (Fig. 57) be R, and the current I. Then the field at the centre, C, is perpendicular to the plane of the coil and is in the direction of the rotational axis of the coil (i.e. vertically downwards).

* For example, Ampère's law of force between current elements: see Maxwell, *Treatise on Electricity and Magnetism* (Vol. II, Part IV, Chapter II).

The field at C due to an element δl of the conductor is

$$\delta B_c = \mu_0 \frac{I\,\delta l}{4\pi R^2},$$

and the total field

$$B_c = \mu_0 \frac{I}{4\pi R^2} \oint dl = \mu_0 \frac{I}{2R}. \tag{11}$$

If the coil has N turns of small section, wound closely together so that the cross-section of the coil may be neglected in comparison with R, each turn contributes an equal amount to B_c. Hence

$$B_c = \mu_0 \frac{IN}{2R}. \tag{11a}$$

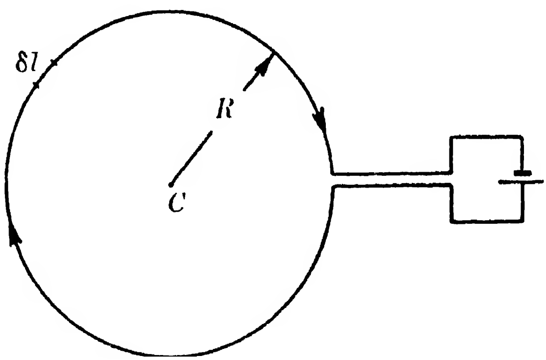


Fig 57. Circular current

9. The magnetic field at any point on the axis of a circular coil.

Let the radius of the coil be R, and let θ be the angle between the axis and the line joining P to a point Q on the coil (Fig. 58).

The field δB_p due to an element δl of a turn is perpendicular to QP, making an angle θ with the plane of the coil, and

$$\delta B_p = \mu_0 \frac{I\,\delta l}{4\pi r^2}.$$

Now δB_p may be resolved into two components, along and normal to the axis of the coil. The normal component will be cancelled by that of another element δl diametrically opposite to Q, but the axial components of all elements will be additive. The axial component of δB_p is equal to

$$\mu_0 \frac{I\,\delta l}{4\pi r^2} \sin\theta.$$

Hence B_p is along the axis of the coil and is given by

$$B_p = \mu_0 \frac{I}{4\pi R^2} \sin^3\theta \oint dl = \mu_0 \frac{I}{2R} \sin^3\theta,$$

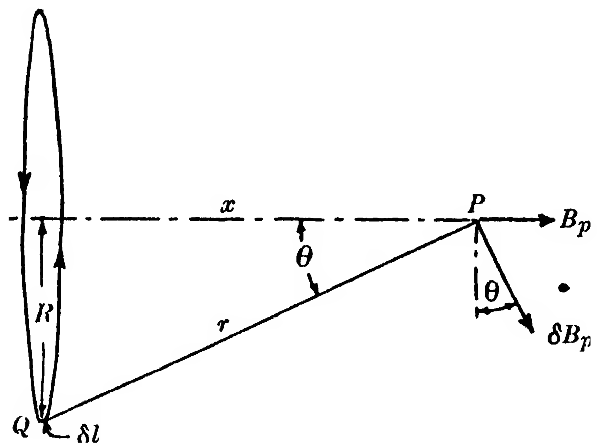


Fig 58 Circular coil

so if there are N concentrated turns of negligible section:

$$B_p = \mu_0 \frac{IN}{2R} \sin^3\theta. \tag{12}$$

If we define the position of P by its distance x from the plane of the coil:

$$B_p = \mu_0 \frac{IN}{2} \frac{R^2}{(R^2+x^2)^{\frac{3}{2}}}. \tag{12a}$$

10. The field at any point on the axis of a short solenoid.

Let the solenoid have radius R, length L, and a single layer of N closely-wound turns (Fig. 59).

In an elementary length δl of the coil, subtending an angle

2θ at the point P (distant x from the centre C) there are $N\,\delta l/L$ turns, and the magnetic flux-density at P due to a current I in this element is, from 3(12),

$$\delta B_p = \mu_0 \frac{IN}{2LR} \sin^3\theta\,\delta l$$

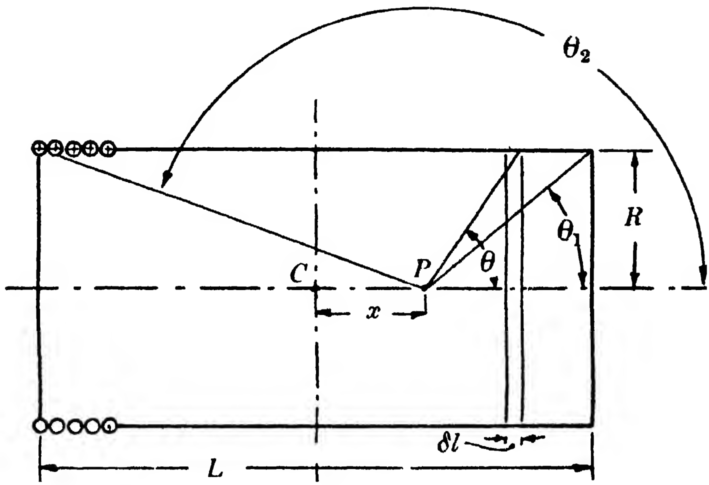


Fig. 59. Short solenoid

From Fig. 60 the length AB

$$= \delta l \sin\theta = \frac{R\,\delta\theta}{\sin\theta},$$

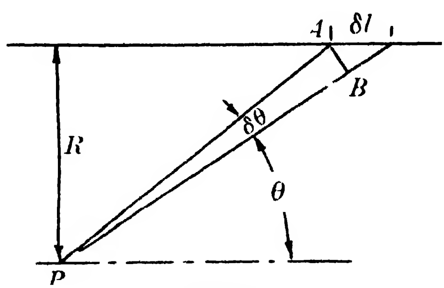


Fig 60

so that $$\sin^2\theta\,\delta l = R\,\delta\theta$$

and $$\delta B_n = \mu_0 \frac{IN}{2L} \sin\theta\,\delta\theta.$$

Hence the field at P due to the complete coil is

$$B_p = \mu_0 \frac{IN}{2L} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

$$= \mu_0 \frac{IN}{2L} (\cos\theta_1 - \cos\theta_2) \qquad (13)$$

$$= \mu_0 \frac{IN}{2L} \left[\frac{\frac{1}{2}L - x}{\sqrt{(\frac{1}{2}L - x)^2 + R^2}} + \frac{\frac{1}{2}L + x}{\sqrt{(\frac{1}{2}L + x)^2 + R^2}} \right] \qquad (14)$$

The field at the centre. Putting $x = 0$ in 3(14):

$$B_c = \mu_0 \frac{IN}{\sqrt{L^2 + 4R^2}}. \qquad (15)$$

The field at the end. Putting $x = \pm \frac{1}{2}L$ in 3(14):

$$B_e = \mu_0 \frac{IN}{2\sqrt{L^2 + R^2}}. \qquad (16)$$

11. The field on the axis of a long solenoid ($L \gg R$).

At the centre. In 3(15) we neglect R^2, so that

$$B_c \doteqdot \mu_0 \frac{IN}{L}. \qquad (17)$$

At the end. Neglecting R^2 in 3(16):

$$B_e \doteqdot \mu_0 \frac{IN}{2L} \doteqdot \tfrac{1}{2} B_c. \qquad (18)$$

12. The field inside an infinitely long solenoid.

Let $n = N/L$, the number of turns per unit length. Then, for any point on the axis, the term $(\cos\theta_1 - \cos\theta_2)$ in 3(13) is now equal to $(\cos\theta - \cos\pi) = 2$, so that

$$B = \mu_0 In. \qquad (19)$$

(Actually this gives the flux-density at *any* point inside the solenoid, not necessarily on the axis. This follows from the circuital law, Section 14.)

13. The field inside a toroidal coil.

An infinitely long solenoid is merely a mathematical abstraction which cannot be constructed. Its peculiar property is that it has no ends, so that every point upon its axis is

similarly situated with respect to the whole coil. Now this condition is also fulfilled if we take a long solenoid and bend it into the form of a closed circle (Fig. 61). Such a coil is termed a *toroid*, and if its radius R is large compared with the radius of the turns (r), we may apply the result obtained for the infinitely long solenoid.

If the total number of turns is N:

$$B=\mu_0 \frac{IN}{L}$$

$$=\mu_0 \frac{IN}{2\pi R}. \qquad (20)$$

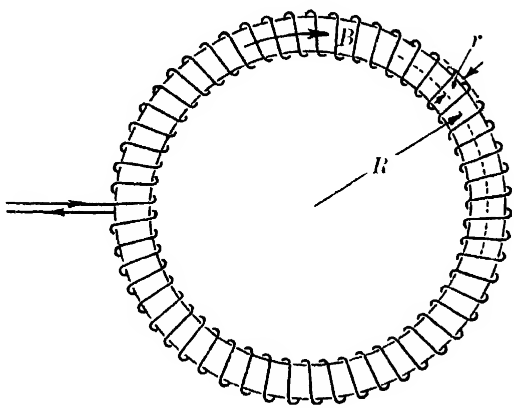


Fig. 61. Toroid

14. The circuital law of the magnetic field.

The value of the flux-density at any particular point near a current-circuit can be obtained by simple calculation in but a few cases, some examples of which have been given above. In the majority of cases the necessary integration becomes difficult or intractable, and for engineering purposes a simpler method of calculation, even though it gives only approximate results, is often of greater practical value. Such a simpler method is provided by the *circuital law* of the magnetic field, which is as follows:

The line-integral of the flux-density B_0 set up by a coil of N turns, carrying a current I, in free space, around any

closed path linking the coil is equal to the current-turns IN multiplied by the constant μ_0,

or: $$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 IN. \tag{21}$$

This law is of great value whenever the *flux-density around a known magnetic circuit is approximately known at all points of a given path*, a condition which is often fulfilled in the iron-cored electro-magnets* used in electro-magnetic machines. The law, coupled with the circuital law for the electric field (equation 2(1)), also enables us to solve the case where both electric and magnetic fields are changing in some region. (See Chapter v, Part I.)

The validity of this law (often called the "work law" in classical theory) is based upon experimental fact: a natural law is always a statement of fact, and every so-called "proof" is merely a derivation of the law from some other law which is itself based upon experiment. For instance, in Chapter II we took the flux-linking law as the experimental starting-point, and derived certain other laws from it. We might just as well have taken one of these other laws (such as that of the force on a charge moving in a magnetic field) as our starting-point and then derived the flux-linking law from it. Our derivations of one law from another make use of a working hypothesis or theory, and the test of our theory lies in the experimental verification of the "laws" we derive.

Consequently the only real "proof" of a natural law is an appeal to experiment, so that no good purpose is really served by "proving" the circuital law (as is done in classical theory) by using the concept of the unit magnetic pole and the magnetic potential,† for the definition of a unit pole is based upon an impossible experiment. If we need the concept of a unit pole, it would be better to define it merely as a mathematical concept analogous mathematically to an electric charge, in terms of the above circuital law, which can, as will be shown, be verified experimentally.

* For the application of the law to such cases, see Chapter IV.

† Nevertheless, the circuital law may be deduced from equation 3(4), but the analysis is beyond the scope of this book.

We shall content ourselves here with a few simple theoretical illustrations of the law, together with a description of a method whereby we may submit it to experimental test.

(*a*) *Illustration of the circuital law for an infinitely long conductor.* The lines of force of the magnetic field of such a conductor are circles concentric with the current, and the magnitude of the field at any point is given by equation 3(8).

The line-integral of B_0 around any closed line of force of radius R is

$$\oint B_0\,dl = 2\pi R B_0 = \mu_0 I.$$

The number of "turns" is unity, so that the law is obeyed in this case for any value of R. Moreover, if we assume the truth of the law for a conductor of appreciable cross-section, we see that equation 3(8) is true for any conductor of circular cross-section, and is not limited to the case of a very fine wire. This fact reminds us of the independence of the value of the electric field, at a point exterior to a uniformly charged sphere, of the size of the sphere.

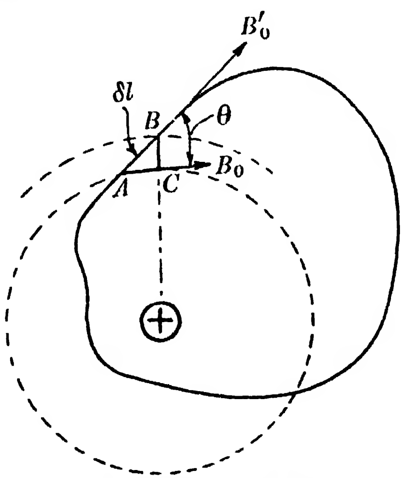


Fig. 62. The circuital law

The result is also true for *any* path linking with the current. Consider Fig. 62, in which the path chosen is of unsymmetrical shape. The increment $\mathbf{B}_0 \cdot \mathbf{dl}$ for the path AB is

$$B_0'(AB = B_0 \cos\theta(AB) = B_0(AC),$$

so that $\oint \mathbf{B}_0 \cdot \mathbf{dl}$ for the complete path is made up of increments $B_0 \delta l$, where δl is measured along a line of force, and increments along paths in a direction in which B_0 has no component. Now since $\oint \mathbf{B}_0 \cdot \mathbf{dl}$ has the same value around any line of force, it follows that, by completing the unsymmetrical path by following a line of force for a short distance AC, and jumping to a new value of R by a radial path such as CB, and so on, the line-integral for a complete path is independent of the path chosen.

(*b*) *For a circular coil of N concentrated turns.* The field at a point P on the axis of the coil (Fig. 58) is

$$B_0 = \mu_0 \frac{IN}{2R} \sin^3 \theta. \tag{3(12)}$$

We shall take the axis of the coil, extending to infinity in both directions, as the closed path around which to evaluate the integral. Hence

$$\begin{aligned} \oint B_0 \, dl &= \mu_0 \frac{IN}{2R} \int_{-\infty}^{+\infty} \sin^3 \theta \, dl \\ &= \mu_0 \frac{IN}{2} \int_0^{\pi} \sin \theta \, d\theta \qquad \text{(see Fig. 60)} \\ &= \mu_0 IN. \end{aligned}$$

It follows from this case that the law must also hold along the infinite axis of any solenoid, for we may form a solenoid by arranging circular coils side by side until the coil has any axial length we may choose. The value of B_0 at any point on the axis is due to the superposition of the flux-densities of all the elementary coils making up the solenoid, and the line-integral for each of these elementary coils is equal to μ_0 times the current-turns of the coil. The line-integral for the complete solenoid must then be equal to the sum of the line-integrals of the elementary coils, or μ_0 times the current-turns of the solenoid.

(*c*) *For a toroidal coil.* The flux-density on the turn axis of a

toroid, whose turn radius is small compared with the ring radius, is

$$B_0 = \mu_0 \frac{IN}{L}, \tag{3(20)}$$

and $\oint B_0\, dl$ around the turn axis

$$= B_0 L = \mu_0 IN.$$

(*d*) *For a moving electric charge; further discussion of displacement current.* The statement of the circuital law presupposes a closed circuit, so that we cannot apply the law to a current element or a moving charge unless we adopt some artifice by which the circuit can be "closed". We do this by means of the displacement current.

Consider a point-charge q, moving with constant velocity v (Fig. 63): provided $v \ll c$, the magnetic field at the point P,

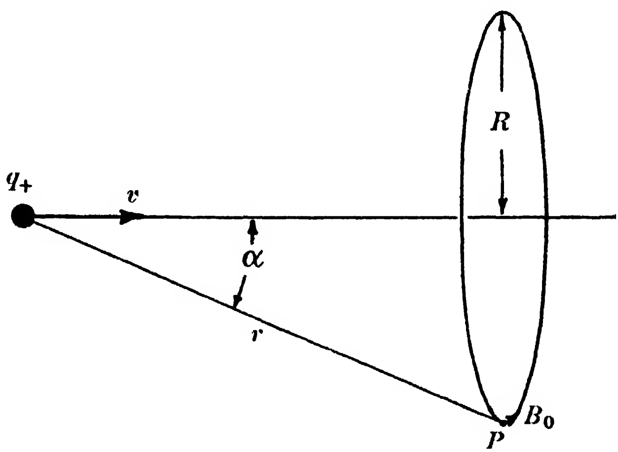


Fig. 63. Moving charge

relative to which the velocity is measured, is

$$B_0 = \mu_0 \frac{qv}{4\pi r^2} \sin \alpha. \tag{3(3)}$$

The field will have this value at every point on a circle through P concentric with the path of the charge. Let us calculate the displacement current through this circle.

The total displacement from the charge q is $\psi = q$, and this is uniformly distributed (unless v is comparable with the

velocity of light) through the total solid angle 4π surrounding the charge. The solid angle included in the cone of semi-angle α is equal to $2\pi(1-\cos\alpha)$, so that the displacement through the circle is given by

$$\psi_\alpha = q\left\{\frac{2\pi(1-\cos\alpha)}{4\pi}\right\} = \frac{q(1-\cos\alpha)}{2}.$$

The displacement current through the circle is

$$I_d = \frac{d\psi}{dt} = \tfrac{1}{2}q\sin\alpha\frac{d\alpha}{dt}$$

but $$\frac{d\alpha}{dt} = \frac{v}{r}\sin\alpha \qquad \text{(see Fig. 64),}$$

hence $$I_d = \frac{qv}{2r}\sin^2\alpha.$$

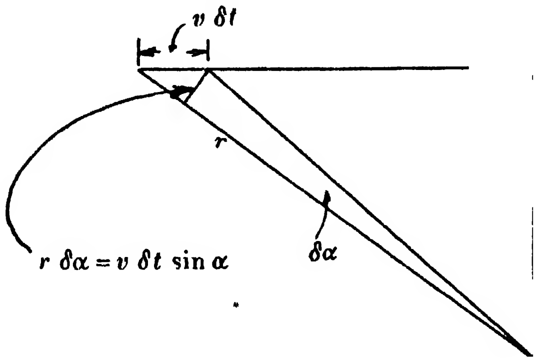


Fig. 64

Now $$\oint B_0\,dl = 2\pi R B_0$$
$$= 2\pi r\sin\alpha\, B_0$$
$$= \mu_0\frac{qv}{2r}\sin^2\alpha$$
$$= \mu_0 I_d.$$

Now what does this result really mean? Does it actually mean that the displacement current, through the circle considered, sets up a magnetic field, whose value at any point on the circle may be calculated by the circuital law? If we calculate the magnetic field of the moving charge in this way, then

clearly we cannot also use equation 3(3), for, if we did, we would credit the magnetic field with having *twice* the magnitude that is consistent with experiment.

The explanation of this apparent anomaly is merely that the magnetic field of a moving charge may be calculated *either* directly from equation 3(3), *or* indirectly by means of the circuital law applied to the displacement current. The two methods are alternative, and must not be used together. In classical theory, it is common to calculate the magnetic field of a moving charge by the second method,* for in that theory the concept of current is not primarily that of moving charges, but of something which causes a magnetic field. We, however, have taken the moving-charge idea as the basis of our *fundamental definition* of electric current, so that, in general, we look on equation 3(3) as the basic equation, and only use the second method (of displacement current) when the *first cannot be applied.*

As an illustration of this use of displacement current, consider the charging condenser of Fig. 16. Using the concept of displacement current, the circuit is closed. The motion of the charges in the connecting wires is expressed in terms of *conduction* current, and their contribution to the magnetic field may be calculated by means of 3(4). But charges are moving *in the condenser plates*, for otherwise the total charge on these plates could not change, and these moving charges must also contribute to the magnetic field. We cannot, however, use 3(4) or 3(3) to calculate their contribution, for we do not know their velocities. But we do know the *rate of change of their electrostatic field* between the plates, and by expressing this as a displacement current, and by use of the circuital law, we now succeed in calculating their contribution, together with that of the conduction current in the wires, to the magnetic field.

The concept of displacement current as applied to the case of a charging condenser, then, does no more than enable us to calculate the magnetic field due to the *velocity* of every moving charge in the circuit. Now the particular importance of displacement current in Maxwell's theory lies in its application

* Originally due to J. J. Thomson: *Phil. Mag.* XI (1881), p. 229.

to the case of electric fields induced by *accelerating charges* (that is, by changing magnetic fields). These induced fields are quite different in their origin from electro-static fields, and it is the supposition that the circuital law is also applicable to their displacement currents which leads to Maxwell's theory of electro-magnetic waves.

(*e*) *Experimental test of the circuital law for conduction currents: the magnetic potentiometer or ampere-turn meter.* Consider a coil of N concentrated turns of area A, placed in a magnetic field whose flux-density has a component B along the axis of the coil.

Let this field collapse to zero in time T. During a short interval δt let the corresponding change in B be δB, so that the flux linking the coil changes by $A\,\delta B = \delta\phi$. An e.m.f. is induced in the coil:

$$e = -N\frac{\delta\phi}{\delta t} = -NA\frac{\delta B}{\delta t}.\text{*}$$

This e.m.f. causes a current given by

$$i = \frac{e}{R},$$

where R is the resistance of the closed circuit of the coil. During the interval δt a quantity δq passes through this circuit, where

$$\delta q = i\,\delta t = \frac{e}{R}\,\delta t = \frac{NA}{R}\,\delta B,$$

so that in time T the quantity flowing through the coil is

$$Q = \int_0^B dq = \left(\frac{NA}{R}\right)B = \frac{N\phi}{R}, \qquad (22)$$

where ϕ is the change in the flux linking the N turns.

If the circuit of the coil is completed through an instrument (such as a flux-meter or ballistic galvanometer) whose deflec-

* The quantity $A\,\delta B$ is the incremental change in the *total* flux linking the coil, which is due in part to the current i flowing in it. This does not affect the result, however, for at $t=0$ and at $t=T$ the current i is zero, so that the flux linking the coil is then due solely to the external field in which it is placed. Thus $A\int_0^B dB = AB$, since the total *change* of flux through the coil due to the current i is zero.

tion gives the quantity of electricity passing through it, this deflection is therefore proportional to the change of flux linkages $N\phi$, and also proportional to the change in the average value, B, of the flux-density over the area A of the search coil. This is the principle of search-coil measurements of magnetic fields, but in the particular application of the principle which we are considering here, the search coil is of a special type.

Consider a long solenoidal coil of axial length L and of small constant cross-section (which may be of any convenient

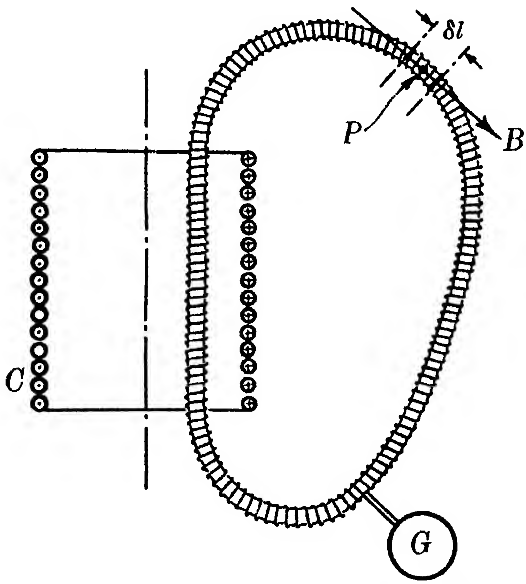


Fig. 65. Ampere-turn meter, or magnetic potentiometer

shape) A. Let this long coil be formed into a closed loop (of any shape) linking with the coil C (Fig. 65) which is to be tested.

Consider a length δl of the looped coil at P, and let the component of flux-density (due to C) at P along the axis of the looped solenoid be B. If the number of turns per unit length of the looped coil be n, the number of turns in the length δl will be $n\delta l$.

Let the current in C be broken, so that B collapses to zero. Then the quantity of electricity passing through the instru-

ment G (in the circuit of the solenoid) due to the collapse of B through the element δl is

$$\delta q = \frac{nA}{R} B\delta l, \qquad \text{from 3(22).}$$

Each element δl of the looped coil will contribute a similar increment δq, so that the total quantity passing through G will be

$$Q = \int dq = \frac{nA}{R} \oint B dl, \tag{23}$$

where the integration is performed around the complete circuit of the looped solenoid.

Now 3(23) shows that the line-integral of the flux-density around any closed path linking a coil may be measured experimentally, and the looped coil may be called a *magnetic potentiometer*, since it measures the difference of magnetic potential, or magneto-motive force (see Section 17), around a closed path linking a current. We shall see later that the magneto-motive force of a coil is proportional to the product (amperes) × (turns), so the looped coil together with the meter G may be called an "ampere-turn meter". The coil may also be used with its ends separated, in which case the difference of magnetic potential between the ends of the coil may be measured, just as the difference of electric potential between two points is measured by means of a voltmeter and connecting leads.

The solenoid may be wound on a flexible leather strap of rectangular section. In order that the closed loop shall not be equivalent to a single turn with respect to a field linking the loop itself, the coil should be wound in two layers in such a way that they magnetize in the same direction along the turn axis, but that any e.m.f.'s induced by a flux *linking the loop* will be equal and opposite in the two layers. The method is shown in Fig. 66, *a*. When formed into a loop, the path of the complete circuit is shown by the arrows in Fig. 66, *b*. Thus there will be no aggregate e.m.f. due to flux linking the loop, the only possible e.m.f.'s being due to flux linking the turns.

The magnetic potentiometer, or ampere-turn meter, gives a simple method of testing the circuital law for any shape of circuit carrying a conduction current. The method of testing

a coil is shown in Fig. 65. The potentiometer strap is looped through the test coil C and the ends fastened together to close the loop. The steady current I in the coil C is measured, and is then broken or reversed by means of a switch in its circuit. The reading of the ballistic galvanometer G is proportional to $\oint B_0 dl$ for the particular path occupied by the loop, and is found to be constant for any path. If, however, the loop is completed *without* linking the coil C, no reading is obtained. For completeness, the experiment should be repeated with straps of various lengths, but with the same number of turns per metre, n.

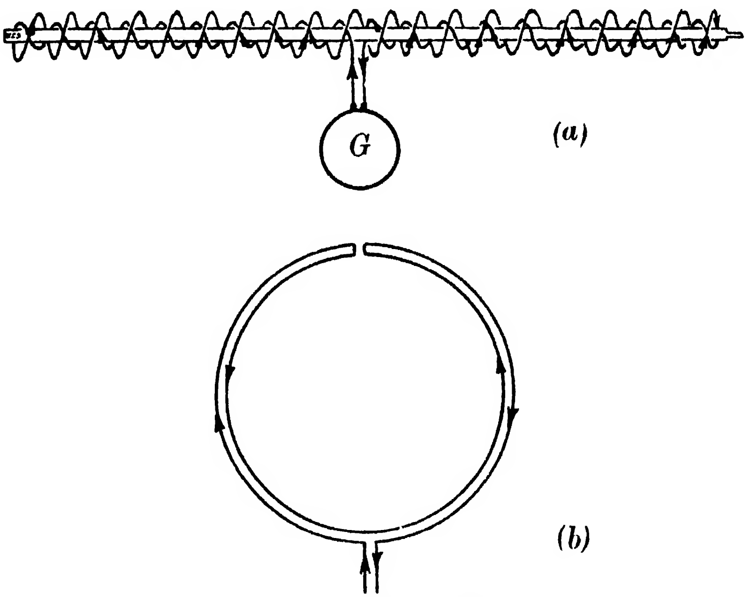


Fig. 66. Method of winding ampere-turn meter

15. General statement of the circuital laws of the electric and magnetic fields.

In the general case of a current-circuit in the neighbourhood of which the electric field $\mathscr{E}$ is not constant, any closed path linking the N turns of the coil will also, in general, surround a displacement current.

The conduction current in the wire itself may not have the

same value at every section of the wire. Let I be the value of the conduction current which is common to all sections, and let the moving charges which are not included in I be allowed for by means of the displacement currents of their electric fields, and let $\frac{d\psi}{dt}$ be the total displacement current linking the closed path. Then the general statement of the circuital law must be (Fig. 67)

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 \left(IN + \frac{d\psi}{dt} \right). \qquad (24)^*$$

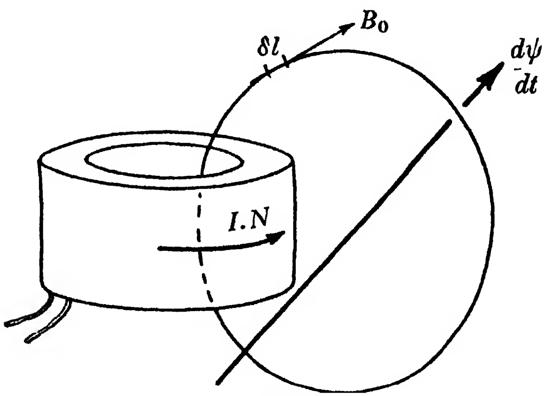


Fig. 67

If a closed path is chosen in a changing electric field which does not also link with the single-valued conduction current I, we have

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 \frac{d\psi}{dt}. \qquad (25)$$

* These expressions are equivalent to the vector form of "Maxwell's Equations":

$$\text{Curl}\,\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$$

and

$$\text{Curl}\,\mathbf{E} = -\dot{\mathbf{B}},$$

respectively

In a material dielectric, $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_d$ (see p. 20), where $\mathbf{E}_0$ in this case is the electro-magnetically induced electric field, and $\mathbf{E}_d$ is the electrostatic field of charges displaced in the dielectric by $\mathbf{E}_0$. Accordingly we have

$$\oint \mathbf{E}_0 \cdot \mathbf{dl} = -\frac{d\phi}{dt} = \oint \mathbf{E}_0 \cdot \mathbf{dl} - \oint \mathbf{E}_0 \cdot \mathbf{dl}$$

$$= \oint \mathbf{E}_0 \cdot \mathbf{dl}, \quad \text{since} \oint \mathbf{E} \cdot \mathbf{dl} = 0,$$

ich is the last term of equation 3(26).

Now, in general, the magnetic field given by equations 3(24) or 3(25) will be variable, so that it must be related to the electric field by the equation

$$\oint \mathbf{E}_0 \cdot \mathbf{dl} = -\frac{d\phi}{dt}, \qquad (26)^*$$

where $\mathscr{E}$ bears a relation to ψ in 3(25).

Hence we shall have an electro-magnetic field whose electric and magnetic components are mutually dependent, the relation between $\mathscr{E}$ and B_0 being theoretically obtained by a simultaneous solution of equations 3(24) and 3(26).

Fortunately, for any particular current-circuit in which the current changes comparatively slowly with time, the term $d\psi/dt$ in 3(24) is negligible compared with the term IN. Consequently for alternating-current circuits for power frequencies (e.g. 50 or 60 cycles per second) we may usually write

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 IN$$

without any appreciable error.

The convenience of this close approximation is considerable. It means that all the formulae obtained for magnetic fields in Sections 4, 8, 9, 10, 11, 12 and 13 of this chapter, although strictly true for steady currents only* (for we neglected any effect of changing electric fields in obtaining them), are also true, within an insignificant error, when I is the instantaneous value of an alternating or changing current, provided that the rate of change is of the order associated with power frequencies.

In radio communication, however, the term $d\psi/dt$ (displacement current) in 3(24) is of prime importance, for on it depends the radiation of energy which provides the communication link. The reason for the use of currents of very high frequency in radio work lies in the necessity for a high value of $d\psi/dt$.

The determination of the fields due to a high-frequency current in a transmitting antenna cannot be obtained by the direct use of equations 3(24) and 3(26), and when we think in terms of electric and magnetic fields the problem appears to

* See footnote on p. 156.

be impossibly complex. Fortunately, however, we can regain simplicity by going back to fundamental ideas about the mutual influence of electric charges, and by so doing we find that the apparent complexity is due to retaining the concept of the magnetic field in studying the effects of *changing* currents as opposed to the effects of *constant* currents. This subject is discussed in more detail in Chapter v, where the concept of the vector potential of the electric current is shown to be the appropriate substitute for the magnetic field in cases of changing currents.

16. The application of the circuital law to two simple cases.

A. *The magnetic field at any point inside a toroidal coil.*

Let the toroid be wound on a non-magnetic ring of uniform section (the shape of this section is immaterial).

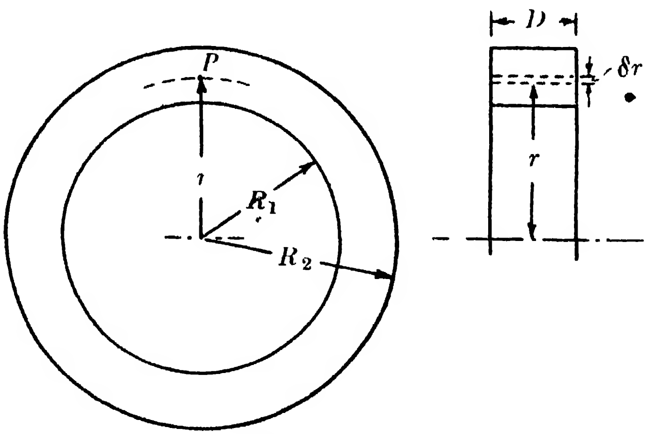


Fig. 68. Toroid

Let B_0 be the flux-density at a point P inside the coil, distant r from the axis of the ring. Then B_0 is the flux-density at all such points, so that

$$\oint B_0\,dl = 2\pi r B_0 = \mu_0 IN,$$

whence
$$B_0 = \mu_0 \frac{IN}{2\pi r}. \tag{27}$$

If the section of the ring is rectangular, it is a simple matter to calculate the total flux set up inside the coil. Let the inner and outer radii of the ring (Fig. 68) be R_1 and R_2, and the axial length D.

The flux-density at any point P at a radius r is given by 3(27) above, so that the flux through an elementary area of radial width δr and length D is

$$\delta\phi = B_0 D\,\delta r,$$

and the total flux through the coil is

$$\phi = \mu_0 IND \int_{R_1}^{R_2} \frac{dr}{2\pi r}$$

$$= \left(\mu_0 D \log_e \frac{R_2}{R_1}\right) \frac{IN}{2\pi}. \tag{28}$$

If the radial thickness of the ring, $(R_2 - R_1)$, is small compared with R_2 the variation of B_0 over the section may be neglected, and B_0 may be obtained from 3(27) by putting $r = \frac{1}{2}(R_1 + R_2)$, whence

$$\phi \doteqdot \mu_0 D \frac{R_2 - R_1}{R_2 + R_1} \frac{IN}{\pi}. \tag{29}$$

If the section of the ring is circular, of radius a, small compared with R, the mean radius of the ring, the flux through the coil is

$$\phi \doteqdot \mu_0 \left(\frac{a^2}{R}\right) \frac{IN}{2}. \tag{30}$$

B. *The field inside a straight conductor of circular section.*

Let the radius of the conductor be R (Fig. 69, a) and let it carry a steady current of I amperes. This current will be uniformly distributed over the cross-section provided the material of the conductor is homogeneous.

Consider a point P inside the conductor distant r from the axis. The current enclosed by a circle of radius r, concentric with the axis of the conductor, is

$$i = I\left(\frac{r}{R}\right)^2.$$

By the circuital law:

$$\oint B_0 dl = \mu_0 i,$$

or

$$2\pi r B_0 = \mu_0 I \frac{r^2}{R^2},$$

so that

$$B_0 = \mu_0 \frac{Ir}{2\pi R^2}. \tag{31}$$

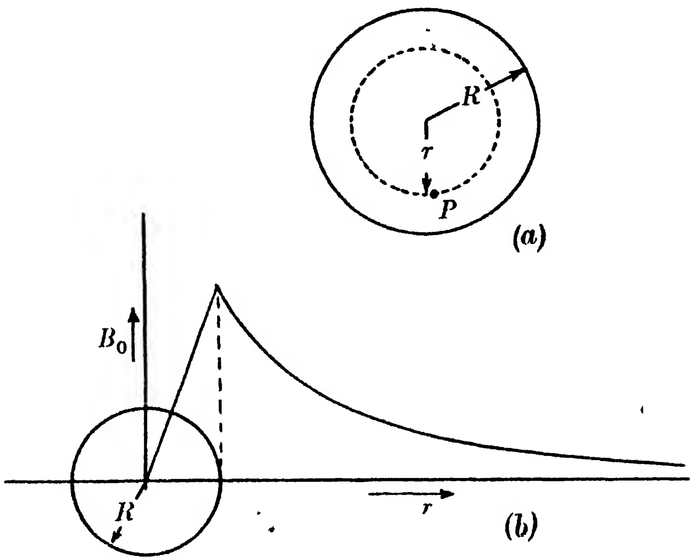


Fig. 69. Field of cylindrical conductor

If the wire consists of material of relative permeability μ (e.g. iron wire) the flux-density at P is $B = \mu B_0$.

The field at a point inside a round conductor is thus proportional to the distance r from the axis, while the field outside (for an infinitely long wire) is inversely proportional to r. The variation of B_0 for a non-magnetic wire is sketched in Fig. 69, *b*.

17. Magneto-motive force (m), and m.m.f. gradient (H).

We are already familiar with the e.m.f. (E) as a total measure of the electric field acting around a closed circuit, the relation

between e.m.f. and resultant field intensity being given by the circuital law, 1(37*c*)

$$\text{e.m.f. } e = \oint \mathbf{E}_0 \cdot \mathbf{dl}.$$

Now since we also have a circuital law for the magnetic field:

$$\mu_0 IN = \oint \mathbf{B}_0 \cdot \mathbf{dl},$$

it is an easy step to invent the term *magneto-motive force* (m.m.f.) to represent the total measure of the magnetizing effect of a coil, around any closed path linking with its turns.

By defining m.m.f. in such a way that it is *mathematically* analogous to e.m.f., we ensure that the laws governing certain magnetic phenomena (such as the law of refraction of a magnetic field, and the law of "energy storage") shall have the same mathematical form as the analogous laws for the electric field. It has already been shown that D (electric) and B (magnetic) are mathematically analogous (see equations 2(12*b*) and 3(7)), as are ϵ_0 and μ_0. Now

$$\text{electro-motive force} = \frac{1}{\epsilon_0}\oint \mathbf{D} \cdot \mathbf{dl} \quad \text{for free space,}$$

where $D/\epsilon_0 = E$, the field intensity. Let us therefore define m.m.f. thus:

$$m = \text{magneto-motive force} = \frac{1}{\mu_0}\oint \mathbf{B}_0\, \mathbf{dl} = IN. \quad (32)$$

The quantity B_0/μ_0 is clearly the *m.m.f. gradient* of the magnetic field, and is mathematically analogous to E. We give it the symbol H, so that

$$\frac{B_0}{\mu_0} = H, \quad \text{and} \quad \text{m.m.f. } m = \oint \mathbf{H} \cdot \mathbf{dl} = IN. \quad (32a)$$

We may now express all the previous expressions for magnetic fields [equations 3(3), 3(4), 3(8), 3(11), etc.] in terms of H

instead of B_0, and by so doing these expressions are simplified by the omission of the factor μ_0. For example, the expression for the field due to a current element $I\,\delta l$, from which we can build up all the others, becomes [see 3(4)]

$$\text{m.m f. gradient } H = \frac{I\,\delta l}{4\pi r^2}\sin\alpha. \tag{33}$$

In conventional theory H is called the *intensity* of the magnetic field, since *magnetic poles*, analogous to electric charges, are accepted as fundamental sources of magnetic fields. The name adopted here, *m.m.f. gradient*, reminds us that even in magnetized matter we recognize *electric currents*, having m.m.f., as the fundamental field-sources.

Units of m and H

Unrationalized c.g.s. units of m.m.f. and H have been in use for so long in conjunction with the practical units of the electric circuit that it is of interest to note the relation between them and the units of these quantities in the rationalized* m.k.s. system.

(A) *Electro-magnetic c.g.s. system.*

The unit of m (magneto-motive force) is the *gilbert.*

The unit of H (m.m.f. gradient) is the *gilbert per cm.* or the *oersted.**

(B) *Rationalized m.k.s. system.*

The unit of m.m.f in the rationalized m.k.s. system is the *ampere-turn*, a unit much used by engineers before the adoption of the r.m.k.s. system, and the corresponding unit of H is the *ampere-turn per metre*. We may note that

$$1 \text{ ampere-turn} = 4\pi \times 10^{-1} \text{ gilbert}$$

and $$1 \text{ ampere-turn per metre} = 4\pi \times 10^{-3} \text{ oersteds}$$

* Adopted by the I.E.C. in 1930.

18. The use of H and B.

H, the m.m.f. gradient of the magnetic field, is mathematically analogous to E, the intensity of the electric field. It might therefore appear that H is the measure of the magnetic field which is most significant, and indeed in the old days of the "material medium" theory H was looked upon as the *stress* in the aether which caused the *strain* B. The main thesis of this book is that we should concentrate our attention upon the behaviour of electric charges (electrons, etc.), and from our point of view the "magnetic field" is merely a useful hypothesis in terms of which we describe the mutual effects of charges in motion, while B becomes a very useful mathematical vector quantity. The force on a charge, moving in a magnetic field, is a simple function of the flux-density B [see equation 2(10)], and this is the fundamental fact underlying the flux-cutting law of induction and the forces experienced by current-carrying conductors. B, then, is the measure of the magnetic field which enters most directly into the laws of electro-magnetic energy conversion which are the very basis of electrical technology.

In the prevalent c.g.s. system, B is numerically equal to H in a non-magnetic medium, and H may be used in formulae without introducing errors into numerical results, as is so often done with total disregard for physical dimensions. The use of an absolute practical-unit system, however (such as the m.k.s. system), forces us to clarify our ideas on the matter, for H is no longer equal numerically to B_0.*

* There is an unfortunate difference in fundamental viewpoint between some electrical engineers and physicists. In the preface to *The Classical Theory of Electricity and Magnetism*, by M. Abraham, we read: "...the distinction in principle between D and E, which is closely connected with the mechanical theory of the aether, has been absolutely abandoned in modern physics, the electro-magnetic conditions at any point in empty space being now regarded as completely defined when we are given *one* electric vector E and *one* magnetic vector B (or H). The numerical identity of E and D (for empty space) in the Gaussian system of units is not, for the physicist, the result of an arbitrary definition, but the expression of the fact that E and D are actually the same thing." The International Electro-Technical Commission, on the other hand, agreed in 1930 that: "for

In the case of the magnetization of iron, which will be studied in Chapter IV, it is more logical to take B_0, the contribution of the magnetizing coil to the total field, as the real cause of the total field B, than to attribute this function to H. Hence we may call B_0 the "magnetizing flux-density", and of course where the substance or medium placed in the field of a coil cannot contribute any field of its own (i.e when it is non-magnetic) we have $B = B_0$, a relation which has none of the mental pitfalls entangled in the statement $B = H$.

The value of the m.m.f. gradient H lies solely in its function as an aid to calculation, especially in the case of simplified "magnetic circuits" containing iron, and in simplifying the expressions for the energy of a magnetic field. It does not possess the direct practical significance of E or B, since it does not give us, directly, the force on an electric charge.

19. The "law" of the magnetic circuit: reluctance and permeance.

The circuital law of the magnetic field may be put into another form which is of practical use in dealing with the "magnetic circuits" of electrical machines, in which the majority of the flux is confined, by the use of iron, within well-defined boundaries. In the absence of iron the only case in which this is true is that of the uniformly wound toroid (Section 16, A, above).

Let the total flux ϕ due to a coil of m.m.f. IN be confined to a path whose limits are well defined, in such a way that the

electrotechnical purposes, the convention should be established that in free space the quantities flux density B and magnetizing force H should be taken as physically different."

This latter statement, apparently, is interpreted by physicists as showing that electrical engineers still believe in the mechanical aether. Some undoubtedly do, but, although from our standpoint the statement of the I.E.C. might have been more happily expressed, the thesis of this book is in no way contrary to the view of modern physicists that the mechanical aether does not exist, while at the same time we maintain that B and H are different. Perhaps we may be forgiven for suggesting that, if E and D (and the same applies to H and B) are *actually the same thing*, then one symbol might, in each case, be sufficient.

flux-density B is everywhere uniform. Let the length of this closed path be L, and its cross-section A. Then

$$\phi = BA = \mu_0 HA \quad \text{(in a non-magnetic medium)}$$

or $$H = \frac{\phi}{\mu_0 A} \quad \text{at all points of the field.}$$

Hence the m.m.f. $m = IN = \oint \mathbf{H} \cdot \mathbf{dl} = HL = \phi\left(\frac{L}{\mu_0 A}\right)$.

We notice that this relation is somewhat analogous to the law of the electric circuit:

$$\text{e.m.f.} = \text{current} \times \text{resistance},$$

so we put

$$\frac{L}{\mu_0 A} = S, \text{ the } \textit{reluctance} \text{ of the magnetic circuit} \quad (34)$$

$$\left(\text{compare} \quad \frac{L}{\gamma A} = R, \text{ the } \textit{resistance} \text{ of a uniform conductor}\right),$$

so that the circuital law, for this special case, may be put

$$\text{m.m.f.} = \text{flux} \times \text{reluctance}. \quad (35)$$

If the path of the flux is in magnetic material of relative permeability μ, the reluctance is then

$$S = \frac{L}{\mu\mu_0 A}. \quad (36)$$

If the path can be split up into lengths, over each of which the flux-density B is uniform, we may calculate the reluctance S_1, S_2, ... of each length and add these together to form the *total* reluctance of the circuit, just as we add the various resistances of sections of an electric circuit, when these are connected in *series*, to obtain the total resistance.

The reciprocal of reluctance has been termed *permeance*, $\mathscr{P}$,

so that $$\mathscr{P} = \frac{\mu\mu_0 A}{L} \quad (37)$$

and $$\text{flux} = (\text{m.m.f.}) \times (\text{permeance})$$

[compare $\quad$ current = (e.m.f.) × (conductance)].

The value of the quantity $\mathscr{P}$ lies in its use in cases where a given flux ϕ splits up into several *parallel* paths, in each of which the flux-density B is sensibly uniform. In this it is comparable to the use of G (conductance) in cases of resistances

in parallel. The resultant permeance of such a *parallel* magnetic circuit is then $\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2 \ldots$.

The concepts of reluctance and permeance are extremely artificial, and it is doubtful whether much is gained by their *quantitative* use. They are, however, of *qualitative* value in obtaining a rough idea of the relative distribution of magnetic flux in a given case, and in discussing qualitatively the effects on the field of making changes in the distribution of iron. In Section 12, Chapter IV, the m.m.f. required for the magnetic field of a machine is calculated by direct use of the circuital law, with no reference to S or $\mathscr{P}$.

The great difficulty in using quantitative values of S and $\mathscr{P}$ lies in the fact that they themselves, in the case of iron, depend upon the flux-density B, so that they cannot be used to determine the flux set up in a given iron circuit by a given m.m.f. In the analogous electrical case, the resistance R is independent of the current, so that the current due to a given e.m.f. can be calculated directly.

20. Self-induced e.m.f.: coefficient of self-inductance.

An e.m.f. is induced in a circuit whenever the magnetic flux linking the circuit changes. Now a current in the circuit is always attended by a magnetic field, which links the circuit, and in the absence of magnetic material this field is strictly proportional to the (steady) current. Thus when the current changes, there must be a change in the linking flux, and an e.m.f. will be induced which, by Lenz's Law, opposes the change of current. This is called the *e.m.f. of self-induction*, and to express it in terms of changing current, instead of in terms of the changing flux, we introduce a constant of the circuit called the *coefficient of self-inductance* (L).

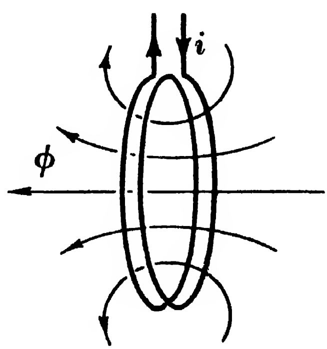


Fig. 70. Self-inductance

Let a coil of N turns carry a current i at time t, and let the flux, due to i, linking the N turns be ϕ (Fig. 70). Then when ϕ changes, an e.m.f. is induced:

$$e = -N\frac{d\phi}{dt} = -N\frac{d\phi}{di}\frac{di}{dt}.$$

If we neglect any displacement currents due to a changing electric field, we may assume that ϕ is proportional to i, so that

$$\frac{d\phi}{di} = \text{a constant,}$$

$$= \text{the linking flux per unit current}$$

and

$$N\frac{d\phi}{di} = \text{a constant,}$$

$$= \text{the flux-turns or flux-linkages per unit current,}$$

$$= L, \text{ the } \textit{coefficient of self-inductance,}^* \qquad (38)$$

so that the self-induced e.m.f. is

$$e = -L\frac{di}{dt}. \qquad (39)$$

The practical unit of self-inductance, corresponding to the volt and the ampere, is the *henry*. In obtaining L, in henries, by means of 3(38), the practical unit of flux (the weber) must be used. If ϕ is in c.g.s. units (maxwells) we have

$$L = N\frac{d\phi}{di} \times 10^{-8} \text{ henries.}$$

Numerical example: the inductance of a toroid. Let us calculate, from the above definition, the self-inductance of a toroid with a ring radius of 20 cm., and a turn radius of 1 cm., uniformly wound with 1000 turns.

Using practical units, the flux linking the coil due to a current of I amperes is

$$\phi \doteqdot \mu_0\left(\frac{a^2}{R}\right)\frac{IN}{2} \text{ webers [3(30)],}$$

so that
$$L = \frac{N\phi}{I} \doteqdot \mu_0\left(\frac{a^2}{R}\right)\frac{N^2}{2}. \qquad (39a)$$

Then
$$\mu_0 = 4\pi \times 10^{-7},$$
$$a = 10^{-2} \text{ metre,}$$
$$R = 0{\cdot}2,$$
$$N = 1000,$$

* This definition of inductance is limited, of course, to cases where we may neglect displacement currents. It is thus a "low frequency" definition.

whence $\qquad L \doteqdot \pi \times 10^{-4}$ henry (H.)

$\qquad\qquad \doteqdot 0{\cdot}314$ milli-henry (mH.).

(*Note:* 1 milli-henry $= 1$ mH. $= 10^{-3}$ henry,
1 micro-henry $= 1\,\mu$H. $= 10^{-6}$ henry.)

21. The energy stored in the magnetic field of a coil of self-inductance L, carrying a current I.

Suppose such a coil has a resistance R and is connected to a source of constant e.m.f., E, at time $t=0$. At time t let the current be i, and during a short interval δt let the current increase by δi. Then we have

$$E - L\frac{\delta i}{\delta t} = iR$$

or
$$E = iR + L\frac{\delta i}{\delta t}.$$

The energy supplied by the source during the interval δt is

$$Ei\,\delta t = Ri^2\delta t + Li\,\delta i.$$

Now the term $Ri^2\delta t$ denotes energy used in overcoming the resistance of the coil, and is converted into heat which cannot be recovered as electrical energy when the current stops. The second term, however, denotes a supply of energy which is recovered as electrical energy when the current falls to zero, since the sign of δi (but not of i) reverses when the current decreases, which means that energy is liberated into the circuit in the form of an e.m.f. supporting the current flow.

Thus energy is *stored* in the circuit in some way when a steady current flows. We may, if we please, think of it as the *energy of motion* of the electric charges and their electric field, the term $Li\,\delta i$ then denoting the energy expended in accelerating the charges (and their electric field) during the interval δt. Now the magnetic field exists at all points in space where the electric field of the moving charges is moving, so that we may think of this kinetic energy as being *stored in the magnetic field.*

If the current rises from zero to a steady value I, the total energy stored in the magnetic field is then

$$W = L\int_0^I i\,di = \tfrac{1}{2}LI^2. \qquad (40)$$

(This expression reminds one of $\frac{1}{2}mv^2$, the kinetic energy of a moving mass.)

Two definitions of the coefficient of self-inductance, L. Equation 3(40) gives us a second method of defining L, and for convenience we give the two alternative definitions here:

(A) L is equal to the product flux × turns, or flux-linkages with the circuit, per unit of current flowing in the circuit.

(B) L is equal to twice the energy stored in the magnetic field of the circuit when unit current flows.

22. The coefficient of mutual inductance, M, of two coils.

Consider two fixed coils (1) and (2), Fig. 71, arranged so that some of the magnetic field of a current i_1 in (1) links with coil (2). Then when i_1 changes an e.m.f. will be induced in (2) by the changing flux. This is called an *e.m.f. of mutual induction.*

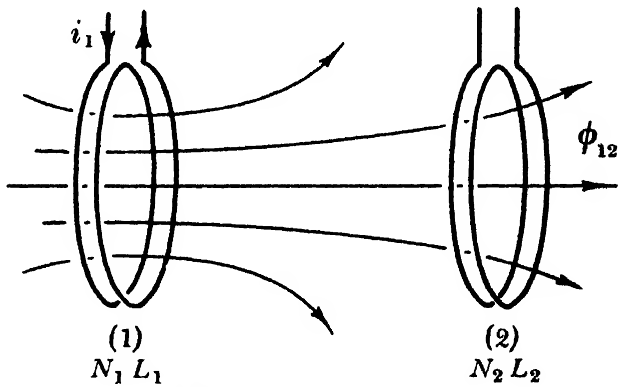


Fig 71. Mutual inductance

Let M_{12} = the flux-linkages with (2) due to unit current in (1). Then if the current in (1) is i_1 the flux linking (2) is given by

$$\phi_{12} = \frac{M_{12} i_1}{N_2}, \quad \text{since} \quad M_{12} = \frac{\phi_{12} N_2}{i_1},$$

so that, when i_1 changes, the e.m.f. induced in (2) will be

$$e_2 = -N_2 \frac{d\phi_{12}}{dt} = -M_{12} \frac{di_1}{dt}. \tag{41}$$

Similarly, if unit current flows in (2), a flux will be produced, some of which will link with (1). Let

M_{21} = the flux-linkages with (1) due to unit current in (2),

then if i_2 changes, there will be an e.m.f. induced in (1):

$$e_1 = -M_{21}\frac{di_2}{dt}. \tag{41a}$$

To prove that $M_{12}=M_{21}$. Now suppose that the current in coil (1) is fully established and let its value be I_1, and let the current in coil (2) be zero. Then there will be energy of amount $\frac{1}{2}LI_1^2$ stored in the magnetic field of the system.

Next let i_2 grow from zero to its final steady value I_2. During its growth suppose that it increases by δi_2 in time δt. Then if the coils magnetize in the same direction, the increase δi_2 will be attended by an increase $\delta\phi_{21}$ in the flux-linking (1) due to i_2, and hence an e.m.f. will be induced in coil (1) *opposing* the current I_1 and given by 3(41*a*) above. In order to maintain I_1 at its constant value, an additional e.m.f. equal to $-e_1$ must be applied in the circuit of coil (1), and in time δt additional energy must be supplied in the same circuit of amount

$$\delta W_m = e_1 I_1 \delta t = M_{21} I_1 \delta i_2,$$

so that, as i_2 rises from zero to I_2 (I_1 being maintained constant), additional energy must be expended in the circuit of (1) of amount

$$W_m = \int_0^{I_2} M_{21} I_1 di_2 = M_{21} I_1 I_2.$$

In addition, energy of amount $\frac{1}{2}L_2I_2^2$ must be expended in the circuit of coil (2) in establishing its magnetic field, and since there is no mechanical work done (the coils are fixed), the total energy stored in the magnetic field of the two coils must be

$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2.$$

Again, suppose that I_2 is fully established and that i_1 now grows from zero to its final value I_1. Then the same reasoning gives us a second expression for the total energy of the magnetic field. That is,

$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2,$$

whence

$$M_{21} = M_{12}.$$

This is a result of great importance; stated in words, it is:

The flux-linkages with coil (2) due to unit current in coil (1)
are equal to
The flux-linkages with coil (1) due to unit current in coil (2).

We now put
$$M_{12} = M_{21} = M,$$
the *coefficient of mutual inductance* of the two coils.

Thus:

DEFINITION OF M.

The mutual inductance of two coils, M, is equal to the flux-linkages with one due to unit current in the other.

The practical unit of mutual inductance is again the *henry*.

23. The total energy stored in the magnetic field of two coils, of self-inductance L_1 and L_2, of mutual inductance M, and carrying currents I_1 and I_2.

From the last section, the energy stored under these conditions, when the coils magnetize in the same direction, is
$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 + M I_1 I_2.$$

If, however, the two coils oppose one another magnetically, the e.m.f. e_1 [equation 3(41 a)] will be in a direction to *support* I_1, so that in order to maintain the latter at its constant value the power input to coil (1) must be *decreased*. Thus the total energy stored is *decreased* by the mutual inductance. In general:
$$W = \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 \pm M I_1 I_2. \tag{42}$$

(The + sign being taken when the coils support, the − sign when they oppose, magnetically.)

24. The circuital property of magnetic flux.

In our elementary discussion of magnetic fields it was stated that the "lines of force" always form closed loops, and since an arbitrary distribution of "lines" can be made to represent, quantitatively, a given flux it follows that any complete section of the "magnetic circuit", or flux-path, should cut a constant amount of flux.

As an illustration, take the case of the magnetic field set up by a current in a single circular loop. Allowing for the finite size of the wire, we can determine the value of B, due to the current, at all points in space and inside the wire. Consider the plane of the coil (z–x–y–z'), Fig. 72, reaching to infinity in all directions. Over a circular area which coincides approximately with the mean area of the coil, and whose edges are given by the points x and y, the field (and the flux) passes from right to left, but outside the limits of this area the flux, spreading theoretically to infinity, passes from left to right. Now if the flux is truly circuital, that passing through the area x–y, from right to left, should be equal exactly to that passing through the rest of the infinite plane, from left to right. By making use of the equality of M_{12} and M_{21} of two coils, deduced in the last section, we can show very simply that this must be so.

Let A and B (Fig. 73) be two coaxial and coplanar circular loops, A being of small radius but B having an extremely large radius. Then if a current flows in B, the field at the centre will approach zero if we imagine that coil B is made larger and larger [see 3(11)]. Hence M, the mutual inductance of the two coils, approaches zero and consequently the net flux linking B due to a current in A must also approach zero, as the radius of B approaches an infinite value.

Now the net flux linking B (due to A) is equal to the algebraic sum of the flux (from right to left) over the area x–y (Fig. 72) and the flux (from left to right) over the area of the infinite plane outside (x–y). Thus these two fluxes are equal. We may also show, in a similar manner, that the total net flux due to A over *any* infinite plane is zero, for we may imagine the plane of the infinite coil to be set at *any* angle to the plane of A, and the mutual inductance, M, will always be zero. Further, the shape of the coil A does not affect the argument. The flux of an electric current, therefore, is a single-valued quantity which may be said to form a closed "magnetic circuit", just as an electric current (including both conduction and displacement currents) has the same value over any complete section of its circuit.

This circuital property of magnetic flux forms the basis of the simple theory by which engineers obtain approximate solutions of the complicated problem of determining the configuration and strength of the magnetic field in electrical machines. It is a property that, all too frequently, is accepted without a proper examination of its credentials. These, as a little consideration of the above argument will show, lie in Faraday's Law of electro-magnetic induction.

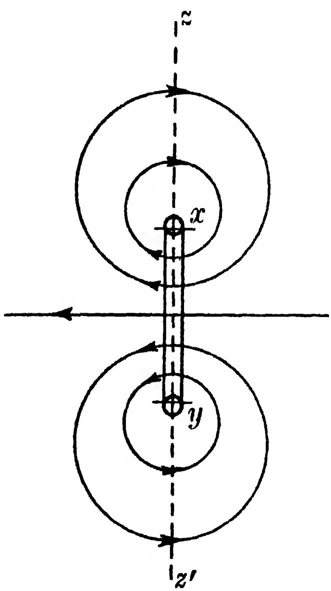


Fig. 72

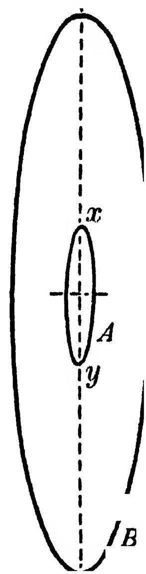


Fig. 73

25. The total self-inductance of two coils of constants L_1, L_2 and M, connected in series.

Let the total self-inductance be L. Then when unit current flows through the circuit the total energy stored in the magnetic field is, from 3(41),

$$W = \tfrac{1}{2}(L_1 + L_2 \pm 2M),$$

which, from the "energy definition" of self-inductance, must be equal to $\tfrac{1}{2}L$. Hence

$$L = L_1 + L_2 \pm 2M. \qquad (43)$$

This relation gives a ready means of measuring the mutual inductance M of two coils, provided a suitable method is available for measuring L.

First measure L with the coils supporting each other magnetically, and let its value be L'. Next reverse the connections of one coil and let the new value of L be L''. Then, from 3(43),

$$L' - L'' = 4M \quad \text{or} \quad M = \tfrac{1}{4}(L' - L''). \tag{44}$$

26. The coefficient of coupling of two coils.

When unit current is flowing in coil (1), Fig. 71, let ϕ_1 be the flux which links its turns N_1 due to this current, and let $k_1\phi_1$ be the portion of this flux which links with the N_2 turns of coil (2).

Similarly, when unit current is flowing in coil (2), let ϕ_2 be the flux which links with its turns N_2 and $k_2\phi_2$ the part of this flux which links with coil (1).

Then
$$L_1 = \phi_1 N_1,$$
$$L_2 = \phi_2 N_2,$$
$$M = k_1\phi_1 N_2 = k_2\phi_2 N_1,$$
whence
$$M^2 = k_1 k_2 L_1 L_2,$$
or
$$M = K_c\sqrt{L_1 L_2}, \tag{45}$$
where $K_c = \sqrt{k_1 k_2}$, the *coefficient of coupling* of the coils.

In practice, k_1 and k_2 must always be less than unity. In the hypothetical case of perfect coupling ($k_1 = k_2 = 1$) we have $M^2 = L_1 L_2$.

27. The mutual force between two coils carrying current.

Let the coils carry steady currents I_1 and I_2, and let them support each other magnetically. Let coil (1), Fig. 74, be fixed, and let coil (2) move, without rotation, through a small distance δx in time δt. Let the corresponding change in M be δM. Then an e.m.f. is induced in (2):

$$e_2 = -I_1 \frac{\delta M}{\delta t}.$$

The additional energy which must be supplied to the circuit of (2) in order to keep I_2 constant is

$$W = -e_2 I_2 \delta t = I_1 I_2 \delta M.$$

Similarly, an e.m.f. $e_1 = -I_2 \dfrac{\delta M}{\delta t}$ is induced in coil (1), so that it also must be supplied with additional energy of amount $I_1 I_2 \delta M$ in order to keep I_1 constant. Thus the total additional

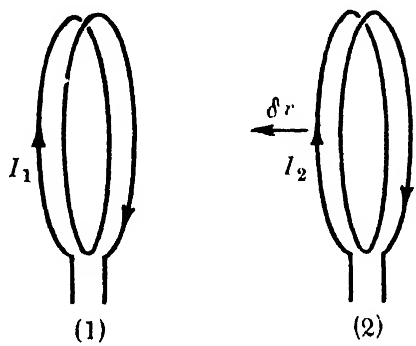


Fig. 74. Force between two coils

energy which must be given to the system in order to maintain I_1 and I_2 constant during the displacement of coil (2) is

$$W = 2 I_1 I_2 \delta M.$$

But, from Section 23, we know that additional energy of amount $I_1 I_2 \delta M$ has been stored in the magnetic field of the system, due to the increase of M, so that the remaining half of the total energy supplied is available for conversion into mechanical work.*

Let F be the force (of attraction in this case) between the two coils. Then

$$F\,\delta x = I_1 I_2 \delta M,$$

so that, in the limit,

$$F = I_1 I_2 \frac{dM}{dx}. \tag{46}$$

Now $I_1 M = N_2 \phi_{12}$, so that

$$F = I_2 N_2 \frac{d\phi_{12}}{dx}.$$

* The mechanical efficiency of the operation is therefore 50 %.

Hence, if a coil of N turns, carrying a current I, is situated in a magnetic field, the component of force in any direction x is given by

$$F = IN\frac{d\phi}{dx}, \tag{2(4)}$$

where ϕ is the total flux linking the coil. Notice that the flux linking the coil due to the current I is constant, and does not contribute to $d\phi/dx$.)

EXERCISE FOR THE STUDENT

If one coil is fixed and the other pivoted so that it may turn about a fixed axis, prove that the torque experienced by it is given by

$$T = I_1 I_2 \frac{dM}{d\theta}, \tag{47}$$

and from this result deduce equation 2(3).

28. The energy stored per unit volume of a uniform magnetic field.

We have already found (Section 21 above) that the energy stored in the magnetic field of a current I flowing in a circuit of self-inductance L is equal to $\frac{1}{2}LI^2$. We can also express the energy per unit volume of a magnetic field in terms of B and H, the simplest method of doing this being to take the case of a toroidal coil in which the radius of the individual turns is small compared with the radius of the ring. The magnetic field is then confined to the space inside the turns, and may be taken as uniform over its section as well as constant around the ring. Let the relative permeability of the ring, on which the coil is wound, be μ.

Let the cross-sectional area of the ring be A and its mean circumference l. Let it have N turns, uniformly wound. Then the energy stored in the field, when a current I flows in the coil, is

$$W = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\phi NI, \tag{48}$$

since

$$L = \frac{\phi N}{I},$$

where ϕ is the total flux of the field.

Now

$$\phi = BA \quad \text{and} \quad NI = HL,$$

whence, substituting these values in 3(48),

$$W = \frac{HB}{2}(AL).$$

But AL is the volume occupied by the magnetic field, so that since the field is uniform we may say that the energy stored per unit volume is

$$W = \frac{HB}{2}. \tag{49}$$

This may also be put:

$$W = \frac{B^2}{2\mu_a} = \frac{H^2\mu_a}{2}, \tag{49a}$$

where $\mu_a = \mu\mu_0$.

In a non-uniform field, we may take indefinitely small elements of volume δV over which the field is sensibly uniform. The total energy of such a field may then be expressed by

$$W = \frac{1}{2}\iiint HB\,dV = \tfrac{1}{2}LI^2. \tag{50}$$

It should be noticed that equations 3(49) and 3(50) are true only if the relative permeability μ is *constant*. This is true for fields in air, but not for fields in ferro-magnetic materials. This may readily be seen by referring to the integration by which we deduced equation 3(40), for we assumed that the inductance L had the same value for all values of the current from zero to the final value I. It will be seen from Chapter IV, Section 5, that when μ is not constant the energy stored per unit volume must be given by

$$W = \int_0^B H\,dB. \tag{51}$$

Equations 3(49) and 3(49a) should be compared with 1(21), which gives the energy stored per unit volume of an electric field.

29. The energy stored in the magnetic field of a moving charged sphere.

An interesting application of equation 3(50) is the calculation of the energy stored in the magnetic field of a charged conducting sphere moving in air or free space. Let the sphere

have radius R and charge q. Then the whole of the space surrounding the sphere can be split up into elementary rings of radius $r\sin\theta$ and sectional area $r\,\delta\theta\,\delta r$. The magnetic field at a point P inside such a ring (Fig. 75) is

$$B=\mu_0\frac{qv}{4\pi r^2}\sin\theta, \qquad 3(3)$$

(provided that $v \ll c$) and has the same value at all points inside the ring.

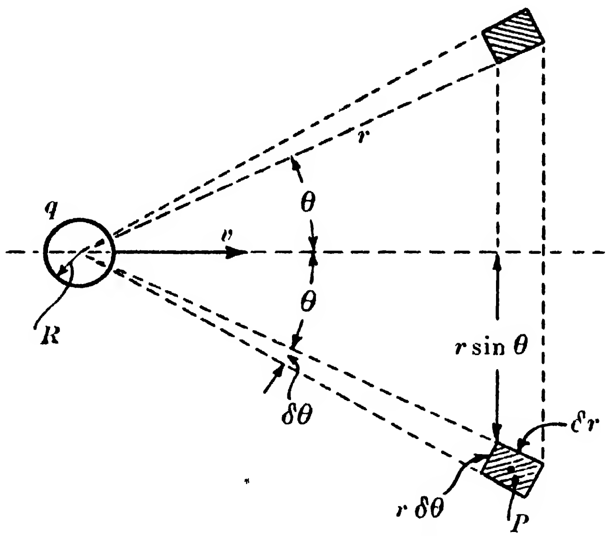


Fig. 75. Moving charged sphere

The volume of the elementary ring is $2\pi r^2\sin\theta\,\delta r\,\delta\theta$, so that the energy stored in the magnetic field in the ring is

$$\delta W=\frac{BH}{2}\times(\text{volume}),\quad \text{where } H=\frac{B}{\mu_0},$$
$$=\mu_0\frac{q^2v^2}{16\pi}\times\frac{\sin^3\theta}{r^2}\,\delta r\,\delta\theta.$$

The total energy stored, in the whole of the space surrounding the charge, is therefore

$$W=\mu_0\frac{q^2v^2}{16\pi}\int_{\theta=0}^{\theta=\pi}\int_{r=R}^{r=\infty}\frac{\sin^3\theta}{q^2}\,dr\,d\theta.$$

Now $$\int_0^\pi \sin^3\theta\,d\theta = -\Big[\cos\theta - \frac{\cos^3\theta}{12\pi}\Big]_0^\pi = \frac{4}{12\pi},$$

so that $$W = \mu_0 \frac{q^2 v^2}{12\pi}\int_R^\infty \frac{dr}{r^2} = \mu_0 \frac{q^2 v^2}{12\pi R}. \tag{52}$$

If this energy is considered to be the kinetic energy due to the motion of the "electro-magnetic mass" of the charge (m_e), we have

$$W = \tfrac{1}{2} m_e v^2 = \mu_0 \frac{q^2 v^2}{12\pi R},$$

or $$m_e = \frac{\mu_0 q^2}{6\pi R}. \tag{53)*$$

Thus the "mass" of the field of a charged sphere is inversely proportional to its radius. According to this theory, a "point" charge is a physical impossibility, since its mass would be infinite.

EXAMPLES, CHAPTER III

1. A motor car is travelling at 300 m.p.h. in a westerly direction (at right angles to the magnetic meridian) at a place where the horizontal component of the earth's magnetic field is 2×10^{-5} weber per sq. metre. Taking the earth as a uniform conducting sphere of radius $6\cdot37 \times 10^6$ metres, with a negative charge of 10^6 coulombs, by what amounts will the vertical potential gradient, E, and the horizontal component of the earth's magnetic field, B, as observed by the driver of the car, differ from the values measured in a stationary laboratory?

   Ans. E will appear to be increased by $2\cdot68 \times 10^{-3}$ volt per metre, while B will appear to be increased by $3\cdot30 \times 10^{-13}$ Wb/m².

2. If the earth's velocity relative to the sun is 3×10^4 metres per sec., and the distance between earth and sun is $1\cdot5 \times 10^{11}$ metres, what would be the value of the magnetic field, observed at the sun, due to the motion of the earth, assuming that it carries a charge of one million coulombs? *Ans.* $1\cdot33 \times 10^{-19}$ Wb/m².

3. A parallel-plate capacitor is constructed with circular plates, between which is a uniform sheet of insulating matérial of dielectric constant K. It is arranged that the metal plates and the insulating

* Equation 3(53) is commonly used to deduce a possible "size" for the electron, using the rather debatable assumption that it is a charged sphere of no internal mass. Using m.k.s. units, we put $\mu_0 = 4\pi \times 10^{-7}$, $q = 1\cdot60 \times 10^{-19}$ coulomb, $m_e = 9\cdot11 \times 10^{-31}$ kilogramme, and from 3(53) we obtain $R = 1\cdot87 \times 10^{-15}$ metre, or $1\cdot87 \times 10^{-13}$ cm., as the "radius" of the electron.

sheet can be rotated about the axis of the discs, either separately or together. When charged to a p.d. V:

A. The metal plates are rotated, the insulator being at rest.

B. The insulating disc is rotated, the metal plates being stationary.

C. Both plates and insulating disc are rotated together.

Show that a magnetic field will exist at a stationary point, its value in case A being proportional to K, in case B proportional to $(K-1)$ and in the opposite direction to that in A, and in case C independent of K and in the same direction as in A.

Note. Consider the electro-static fields E_0' and E_a' of the charges on the plates, and the induced charges on the insulator, respectively, and refer to Section 2 of this chapter.)

4. A very long, straight, thin copper ribbon of width d carries a steady current of I amperes. Find an expression for the flux-density at a point in the plane of the ribbon and distant a from its nearer edge.

Solution. Consider the field at P (Fig. 76) due to the current in an

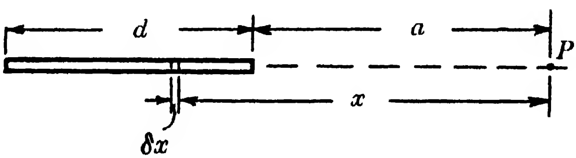


Fig. 76

elementary filament of the conductor of width δx and distant x from P. The current in this filament is

$$\delta I = \frac{I}{d}\delta x,$$

so that the field at P due to the filament is (equation 3(8))

$$\delta B_0 = \mu_0 \frac{I}{2\pi d}\frac{\delta x}{x},$$

whence the field due to the whole current I is

$$B_0 = \mu_0 \frac{I}{2\pi d}\int_a^{a+d}\frac{dx}{x} = \mu_0 \frac{I}{2\pi d}\log_e \frac{a+d}{a}.$$

5. Show that the magnetic field at a point distant a from the centre of a single circular turn of radius R, in the plane of the turn, is given by

$$B_0 = \mu_0 \frac{IR}{\pi(R^2-a^2)}\int_0^{\frac{1}{2}\pi}\sqrt{1-\left(\frac{a}{R}\right)^2 \sin^2\phi}\, d\phi,$$

for points inside the turn, and plot a curve showing the variation of B_0 over a diameter of the circle.*

* See A. Russell, "The Mathematical Theory of the Magnetic Field round a Circular Current, and Allied Problems", *Jl. I.E.E.* LXVII (1929), p. 655.

Solution. Consider the field at P due to the current I in a length δl of the wire between Q and Q'. Let $PQ = D$ and let PN be the normal to the tangent at Q (Fig. 77), then

$$\delta B_0 = \mu_0 \frac{I\,\delta l}{4\pi D^2}\sin\alpha \qquad 3(4).$$

Now the area of the triangle

$$PQQ' = \tfrac{1}{2}D^2\,\delta\phi = \tfrac{1}{2}PN\,\delta l = \tfrac{1}{2}D\sin\alpha\,\delta l,$$

so that
$$\sin\alpha\,\delta l = D\,\delta\phi$$

and
$$B_0 = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{d\phi}{D}.$$

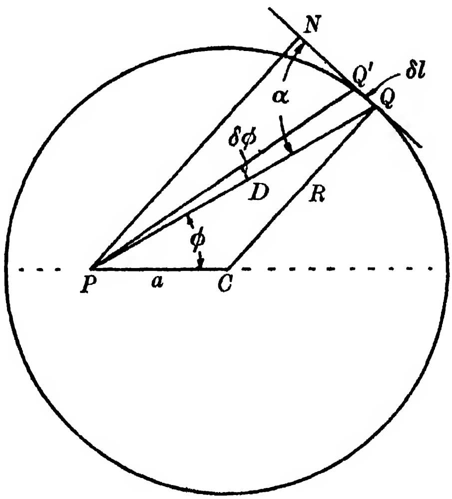


Fig. 77

Now
$$D = PQ = a\cos\phi + \sqrt{R^2 - a^2\sin^2\phi}$$

and
$$\frac{1}{D} = \frac{-a\cos\phi + \sqrt{R^2 - a^2\sin^2\phi}}{R^2 - a^2},$$

whence
$$B_0 = \frac{\mu_0 I}{4\pi(R^2 - a^2)}\int_0^{2\pi}\sqrt{R^2 - a^2\sin^2\phi}\,d\phi \quad \left(\text{since } \int_0^{2\pi}\cos\phi\,d\phi = 0\right)$$
$$= \frac{\mu_0 IR}{\pi(R^2 - a^2)}\int_0^{\frac{1}{2}\pi}\sqrt{1 - k^2\sin^2\phi}\,d\phi,$$

where $k = a/R$.

The integral may be evaluated by first expanding the expression by the Binomial Theorem. It is, however, the "standard elliptic integral" denoted by "E", values of which are to be found in any good set of mathematical tables (e.g. Dale's).

In order to plot a curve showing the variation of B_0 over a diameter, take the simple case where $R = I =$ unity. Thus

$$B_0 = \frac{\mu_0}{\pi(1-k^2)}(E),$$

where k is the distance of P from the centre C.

The value of the integral (E) is usually given in tables in terms of an angle θ, where $k = \sin\theta$, the *modulus* of the integral. We may tabulate the results as follows, for various values of θ:

$\theta°$	E	$k = \sin\theta$	$(1-k^2)$	$B_0/4\mu_0$ $[E/(1-k^2)]$	Relative values
0	1·5708	0·0 (centre)	1·000	1·571 ($\frac{1}{2}\pi$)	1·000
10	1·5590	0·1736	0·970	1·607	1·023
20	1·5238	0·3420	0·883	1·727	1·100
30	1·4675	0·5000	0·750	1·958	1·247
40	1·3931	0·6428	0·587	2·37	1·51
50	1·3055	0·7661	0·414	3·15	2·00
60	1·2110	0·8660	0·250	4·84	3·09
70	1·1184	0·9397	0·117	9·56	6·1
80	1·0401	0·9848	0·030	34·45	22·0
90	1·0000	1·0000	0·000	Infinite	Infinite

Thus up to a radius of 17·4 % of the radius of the loop the magnetic field in the plane of the loop does not vary more than 2·3 %. The infinite value at $k = 1$ is of course never reached owing to the finite diameter of the wire. The variation over a diameter is shown graphically in Fig. 78.

6. A concentrated coil of N turns of radius R carries a current I_1. At a point on its axis, and distant x from its plane, is situated a coaxial concentrated coil of n turns, radius r and carrying a current I_2. Obtain an expression for the mutual force between the coils, given that r is small so that the variation of field strength across the plane of the second coil (due to I_1) may be neglected. Show that this force is a maximum when x is equal to $\frac{1}{2}R$. (Cambridge, A, 1924.)

Solution. The force between the coils may be obtained by using either equation 2(4) or 3(46). We shall use the latter, since by so doing we include the calculation of the mutual inductance.

Let B_0 be the flux-density at P, the centre of the small coil, of the field due to the large coil (Fig. 79). Then the flux linking the small coil owing to I_1 in the first is

$$\phi_{21} = \pi r^2 B_0,$$

and the mutual inductance M is

$$M = \frac{\phi_{21} n}{I_1} = \frac{\pi r^2 B_0 n}{I_1}.$$

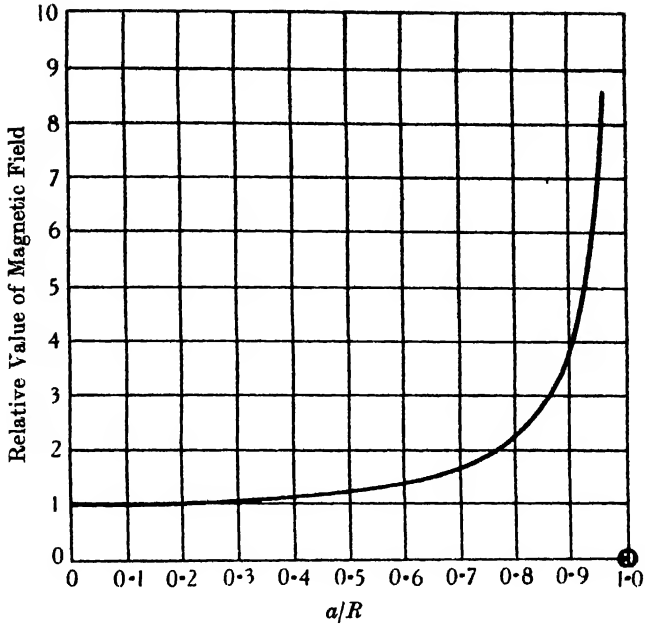


Fig. 78. Variation of magnetic field over plane of circular loop

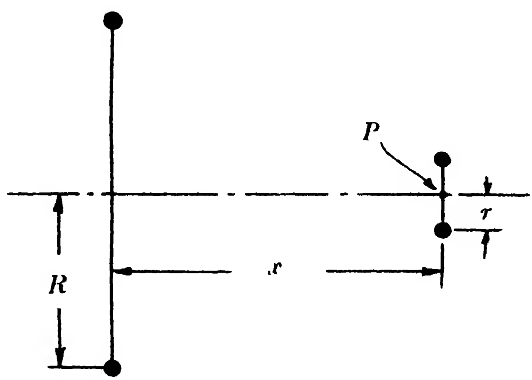


Fig. 79

From 3(12a): $$B_0 = \mu_0 \frac{I_1 N}{2} \frac{R^2}{(R^2+x^2)^{\frac{3}{2}}},$$

so that $$M = \frac{\mu_0}{2}(\pi r^2 R^2 n N)(R^2+x^2)^{-\frac{3}{2}}$$ $$= K(R^2+x^2)^{-\frac{3}{2}},$$

where $$K = \frac{\mu_0}{2}(\pi r^2 R^2 n N)$$

From 3(46) the mutual force is

$$F = I_1 I_2 \frac{dM}{dx},$$

$$\frac{dM}{dx} = -3Kx(R^2+x^2)^{-\frac{5}{2}},$$

so that
$$F = 3KI_1 I_2 \frac{x}{(R^2+x^2)^{\frac{5}{2}}}.$$

This force is zero both when $x=0$ and when x is infinite. At some intermediate value of x it must pass through a maximum. This will be when $dF/dx=0$, or

$$\frac{1}{(R^2+x^2)^{\frac{5}{2}}} - \frac{5x^2}{(R^2+x^2)^{\frac{7}{2}}} = 0,$$

i.e. when
$$R^2+x^2-5x^2 = 0,$$

or
$$x = \tfrac{1}{2}R.$$

(It is interesting to note that if we endeavoured to calculate M in this problem by calculating the flux linking the large coil due to one ampere in the small one, we would experience some difficulty.)

7. A single turn of wire forms the perimeter of a square of side $2a$ and carries a current I. Show that the magnetic force due to this circuit at a point on the central normal to its plane and distant x from the plane is

$$\frac{8a^2 I}{(a^2+x^2)\sqrt{2a^2+x^2}}.$$

A non-metallic former of length l has a square cross-section of side d. It is closely and uniformly wound throughout its length with T turns (total) of negligible thickness. Show that when a current I amperes flows in the winding, the magnetic force at the centre of the former is

$$\frac{1{\cdot}6(IT)}{l}\tan^{-1}\frac{l}{\sqrt{l^2+2d^2}}. \qquad \text{(Oxford, 1935).}$$

Part 1. The flux-density at P (Fig. 80) due to one side AB of the square is (see Section 5 of this chapter)

$$B' = \frac{\mu_0 I}{4\pi PM}\int_{\theta}^{\pi-\theta} \sin\alpha\, d\alpha.$$

Now $PM = \sqrt{a^2+x^2}$, and $\cos\theta = \dfrac{a}{\sqrt{2a^2+x^2}} = -\cos(\pi-\theta)$

whence
$$B' = \frac{\mu_0 a I}{2\pi\sqrt{a^2+x^2}\sqrt{2a^2+x^2}}$$

in a direction making an angle β with the normal to the axis of the coil.

B' may be resolved along and normal to this axis, the normal component being cancelled by that due to the opposite side. Thus the field at P due to all four sides will be along the axis and equal to four times the axial component of B'. That is

$$B = 4B'\sin\beta, \quad \text{where} \quad \sin\beta = \frac{a}{\sqrt{a^2+x^2}},$$

$$= \frac{2\mu_0 a^2 I}{\pi(a^2+x^2)\sqrt{2a^2+x^2}}.$$

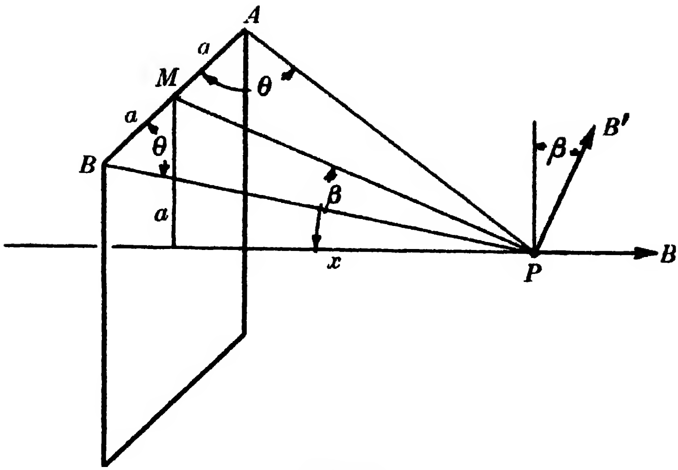


Fig. 80

Part 2. This part of the problem is most readily solved by putting the expression for B in terms of the angle β.

Now
$$(a^2+x^2) = a^2 \operatorname{cosec}^2 \beta$$
and
$$(2a^2+x^2) = \left(\frac{a^2}{a^2+x^2}+1\right)(a^2+x^2),$$
so that
$$\sqrt{2a^2+x^2} = \frac{a\sqrt{\sin^2\beta+1}}{\sin\beta}$$
and
$$B=\mu_0\frac{2I\sin^3\beta}{\pi a\sqrt{\sin^2\beta+1}}.$$

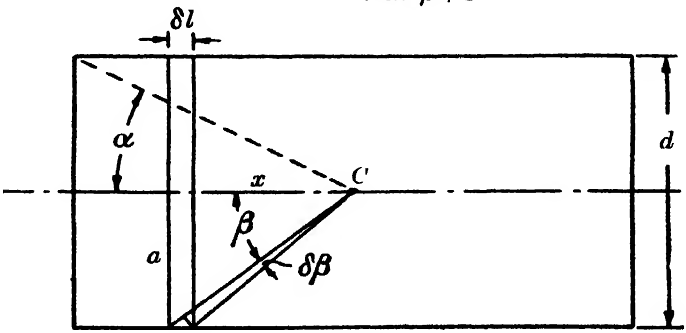


Fig. 81

Consider the field at the centre of the coil, C, due to a length δl of the coil (Fig. 81). The number of turns in this element is $\frac{T\,\delta l}{l}$, so that the field at C due to the whole coil is
$$B_c=\mu_0\int_0^l \frac{2IT\sin^3\beta}{\pi a l\sqrt{\sin^2\beta+1}}\,dl;$$

but $$\delta l \sin\beta = a \operatorname{cosec}\beta\, \delta\beta \quad \text{or} \quad \delta l = \frac{a\,\delta\beta}{\sin^2\beta},$$

so that
$$B_c = \frac{\mu_0 2IT}{\pi l}\int_\alpha^{\pi-\alpha} \frac{\sin\beta\, d\beta}{\sqrt{2-\cos^2\beta}}$$

$$= \frac{\mu_0 2IT}{\pi l}\left[{}_{\alpha}^{\pi-\alpha} \sin^{-1}\left(\frac{\cos\beta}{\sqrt{2}}\right)\right]$$

$$= \frac{\mu_0 4IT}{\pi l}\sin^{-1}\frac{l}{\sqrt{2(l^2+d^2)}}$$

$$= \frac{\mu_0 4IT}{\pi l}\tan^{-1}\frac{l}{\sqrt{l^2+2d^2}}.$$

(*Note*. If the expression for B in terms of x is retained, integrate by making the substitution: $z = \dfrac{x}{\sqrt{\frac{1}{2}d^2 + x^2}}$.)

8. Two square coils, of sides a and b, and of concentrated turns n_1 and n_2, are arranged coaxially with their planes parallel and a distance x apart. If b is very much smaller than a, find an expression for the coefficient of mutual inductance, M, and show that the force between them, when carrying currents I_1 and I_2, is given by

$$F = 2MI_1I_2\frac{x(5a^2+12x^2)}{(a^2+4x^2)(a^2+2x^2)}.$$

9. A solenoid consisting of 1000 turns of wire has a diameter of 20 cm. and a length of 40 cm. Situated coaxially with the solenoid and at its centre is a small coil, consisting of 100 turns of fine wire, the diameter of the coil being 4 cm. and its resistance 50 ohms. A current of 5 amperes is flowing in the outer solenoid, and this current is reduced to 1 ampere at a steady rate, the time occupied in the reduction being $\frac{1}{2}$ sec. Calculate the magnitude of the current induced in the small coil.

(Cambridge, A, 1915.)

Solution. The field at the centre of the outer solenoid is given by 3(15), where

$$N = 1000, \quad L = 0{\cdot}4 \text{ metre}, \quad R = 0{\cdot}1 \text{ metre}, \quad \mu_0 = 4\pi \times 10^{-7},$$

whence $$B_c = (2{\cdot}81 \times 10^{-3})\, I \text{ Wb/m}^2.$$

The change of flux linking the small coil (assuming B to be uniform over its area) when the current falls by 4 amperes is

$$\phi = \pi r^2 B_c = \pi(0{\cdot}02)^2 \times 2{\cdot}81 \times 10^{-3} \times 4 = 1{\cdot}41 \times 10^{-5} \text{ weber.}$$

From 3(22), the quantity which passes through the small coil due to the induced e.m.f. is

$$Q = \frac{N\phi}{R} = \frac{100 \times 1{\cdot}41 \times 10^{-5}}{50} = 2{\cdot}82 \times 10^{-5} \text{ coulomb.}$$

The time taken for the passage of this charge is 0·5 sec., so that the average value of the current in the coil is

$$I = \frac{Q}{T} = 5{\cdot}64 \times 10^{-5} \text{ ampere.}$$

10. Deduce equation 3(8), for the magnetic field of an infinitely long wire, by considering the moving electric field of the moving electrons in the wire, obtaining the magnetic field from equation 3(6). (Take q as the moving charge per unit length of the wire, and see the foot-note to Section 4 on p. 132.)

11. Prove that the m.m.f. along the axis of a short solenoid from end to end (i.e. the difference of magnetic potential between its ends) is

$$IN\left\{\sqrt{\frac{R^2}{L^2}+1}-\frac{R}{L}\right\}.$$

12. Describe the phenomenon known as "pinch effect" exhibited in a conductor carrying current. Deduce an expression for the magnitude of this effect, and explain how it has been applied to the measurement of large alternating currents.

If a cylindrical column of mercury 1 cm. diameter carries a current of 100 amperes, calculate the intensity of the mechanical pressure due to the pinch effect: (1) at a radius of 0·25 cm., (2) at the axis of the conductor. Find also the total axial mechanical force arising from this effect. (London, External B.Sc., 1934.)

Solution. Consider a conductor of circular section (Fig. 82), carrying a steady current I. By equation 3(31) the flux-density inside the conductor, at a point distant r from the axis, is

$$B_r = \frac{\mu\mu_0 I r}{2\pi R^2},$$

where μ is the relative permeability of the wire.

The moving charges (which constitute the current) at this point will be moving in this magnetic field,* and will experience a force directed towards the axis of the wire. This force is transmitted to the material structure of the wire in such a way as to tend to decrease its section. It is therefore named the "pinch effect", and is best shown experimentally when the conductor is liquid, e.g. a column of mercury.

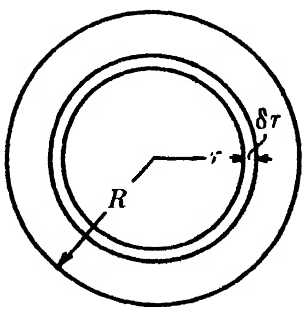


Fig. 82. Pinch effect

To calculate the effect, consider the total inward force on the current flowing in an elementary cylinder of radius r and thickness δr. The current in this element is

$$\delta I = \frac{2\pi r\,\delta r}{\pi R^2} I = \frac{2rI}{R^2}\,\delta r.$$

* See also Section 7.

The force on this current (radially inwards) is equal to $B_r\,\delta I$ per unit of axial length, and this acts uniformly over a cylindrical surface of area $2\pi r$. Thus the inward pressure due to the element is

$$\delta p = \frac{B_r\,\delta I}{2\pi r} = \frac{\mu\mu_0 I^2}{2\pi^2 R^4} r\,\delta r,$$

and the total pressure at a point distant a from the axis will be

$$p_a = \int_0^R dp = \frac{I^2\mu\mu_0}{4\pi^2 R^4}(R^2 - a^2). \qquad (54)$$

In a column of mercury this acts as a hydrostatic pressure; i.e. it acts equally in all directions at any point. If the column is in contact at both ends with metal plates, these plates will experience a force due to the axial pressure.

The total axial force acting upon the cross-section of the elementary cylinder is

$$\delta F = p_r\,2\pi r\,\delta r = \frac{\mu\mu_0 I^2}{2\pi R^4}(R^2 r - r^3)\,\delta r.$$

The total axial force over the complete cross-section is then

$$F = \int_{r=0}^{r=R} dF = \frac{\mu\mu_0 I^2}{8\pi}. \qquad (55)$$

In the case of the mercury column in this problem:

$R = 5\times 10^{-3}$ metre, $I = 100$ amperes, $\mu = 1$, $\mu_0 = 4\pi\times 10^{-7}$.

(1) At $a = 2{\cdot}5\times 10^{-3}$ metre, from equation 3(54),

$$p_a = \frac{30}{\pi} \text{ newtons per sq. metre}$$
$$= \frac{300}{\pi} \text{ dynes per sq. cm.}$$
$$= 0{\cdot}0975 \text{ gm. per sq. cm.} = 0{\cdot}00138 \text{ lb. per sq. in.}$$

(2) At $a = 0$ (at the axis),

$$p_a = \frac{40}{\pi} \text{ newtons per sq. metre.}$$

(3) The total axial force [equation 3(55)] is

$$F = 5\times 10^{-4} \text{ newtons}$$
$$= 50 \text{ dynes.}$$

(For the application of the "pinch effect" to the measurement of large alternating currents see E. F. Northrup, "Some newly observed manifestations of forces in the interior of an electric conductor", *Phys. Rev.* XXIV (1907), pp. 474–97.)

13. Two straight parallel conductors, each carrying a current I, are of length L and distant d apart, such that the lines joining their neighbouring ends are at right angles to their lengths. Prove that the mutual force between the conductors is

$$F = \frac{\mu_0 I^2}{2\pi d}(\sqrt{L^2 + d^2} - d).$$

14. A circular coil of N concentrated turns of radius R has a small circular coil, of n concentrated turns of radius r, placed at its centre so that the two coils have a common diameter and the angle between their planes is θ. If $r \ll R$, show that the coefficient of mutual inductance is given by

$$M = \frac{\mu_0 \pi r^2 N n}{2R} \cos\theta.$$

If the coils carry currents I_1 and I_2, what is the mutual torque when $\theta = 90°$?

15. In a Helmholtz galvanometer two similar circular coils of radius R and N concentrated turns are placed coaxially with their planes a distance R apart. Show that the magnetic field at a point on the axis, midway between the coils, when each carries a current I in such a direction that they support each other magnetically, is given by

$$B_0 = \mu_0 \frac{8IN}{5\sqrt{5}R}.$$

If $R = 0{\cdot}1$ metre, $N = 10$, and $I = 1$, what is B_0 at this point in gauss?

Ans. 0·897 gauss.

16. Plot curves showing the variation of the flux-density along the axis of

(a) A single circular loop of radius R, for points on the axis between the plane of the coil and a point distant $5R$ from it.

(b) A solenoid of radius R and length $10R$, for points on the axis between the centre of the solenoid and a point distant $10R$ from the centre.

17. A closed rectangular loop of wire has sides of lengths $2a$ and $2b$ cm. Show that, due to a current I (ab-amperes) flowing in the loop, the magnetic force at the centre will be

$$H = \frac{4\sqrt{a^2+b^2}\, I}{ab} \text{ oersteds.}$$

A ring of non-magnetic material has a rectangular section, the internal and external radii being r_1 and r_2 cm. and the thickness, parallel to the axis of the ring, d cm. It is closely and uniformly wound with T turns of fine wire. Show that the total flux in the ring per ampere in the winding is

$$\frac{Td}{5} \log_e \frac{r_2}{r_1} \text{ maxwells,}$$

and state the inductance of the winding in henries. The space occupied by the wire may be neglected. (Cambridge, A, 1934.)

Ans. $L = \dfrac{T^2 d}{5} \log_e \dfrac{r_2}{r_1} \times 10^{-8}$ henries.

18. A long straight copper wire has a circular section and carries a steady current I amperes. Assuming that the only appreciable field within the wire is that due to the current in the wire itself, show that

the energy stored in the magnetic field inside the wire is given by $I^2/400$ ergs per cm. length. (Cambridge, A, 1933.)

(*Note.* Use equations 3(31) and 3(49) and obtain the result: $\mu_0 I^2/4$ per unit length of the wire; then convert units.)

19. A return circuit consists of two concentric copper tubes (Fig. 83). A steady current of I amperes flows, in opposite directions, in each tube in an axial direction. What is the magnetic field at any point P, in air, at a distance r from the axis? (Use the circuital law.)

Ans. $B_0 = 0$ inside the inner tube,

$$B_0 = \frac{\mu_0 I}{2\pi r} \text{ in the space between the tubes,}$$

$B_0 = 0$ outside the outer tube.

20. In Ex. 19 the "concentric main" forms a closed circuit of axial length L. A single wire situated along the axis (A, Fig. 83) forms a closed circuit of the same length. If each tubular conductor now carries a changing current whose value at any instant is i (in opposite directions in the two tubes), show that an e.m.f. is induced in the axial wire of total value

$$e = -\frac{\mu_0 L}{2\pi} \log_e \frac{R_2}{R_1} \frac{di}{dt}.$$

(Find the total flux in the annular space between the two tubes: this flux links with the circuit of the axial conductor.)

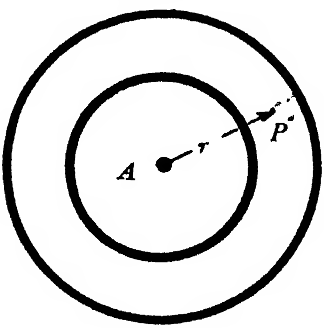


Fig. 83

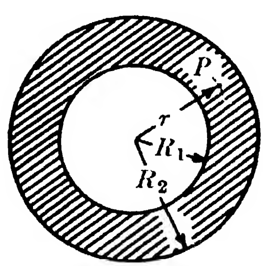


Fig. 84

21. A long straight copper tube (Fig. 84), of inner and outer radii R_1 and R_2, carries a steady current I. Show that the magnetic field at a point P, distant r from the axis of the tube and inside the copper wall, is given by

$$B_0 = \frac{\mu_0 I}{2\pi} \frac{(r^2 - R_1^2)}{r(R_2^2 - R_1^2)}.$$

If the return path of the current I is a conductor inside the copper tube, show that the magnetic field at the point P is given by

$$B_0 = \frac{\mu_0 I}{2\pi} \frac{(R_2^2 - r^2)}{r(R_2^2 - R_1^2)}.$$

22. A "concentric cable" consists of a solid round conductor of radius a with a surrounding concentric cylindrical sheath of inner and outer radii b and c (Fig. 85). Equal currents flow, in opposite directions, in the core and sheath. By finding the total energy stored in the magnetic field when the value of these currents is unity, show that the coefficient of self-inductance, per metre length of the cable, is

$$L = \frac{\mu_0}{4\pi}\left[2\log_e\frac{b}{a} + \frac{1}{2} + \frac{1}{(c^2-b^2)}\left\{\frac{2c^4}{(c^2-b^2)}\log_e\frac{c}{b} - \frac{3c^2-b^2}{2}\right\}\right].$$

(*Note.* The results of Exs. 18, 19 and 21 are all helpful in this case.

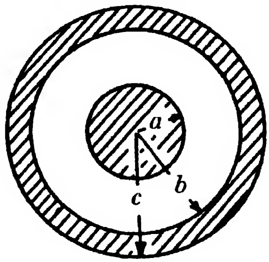


Fig. 85. Concentric cable

23. Two coils, A and B, are arranged so that:

A current in A sets up a flux, 80 % of which links with B.

A current in B sets up a flux, 75 % of which links with A.

When the coils are connected in series to support each other magnetically, the total self-inductance is 5 milli-henries. When the coils are connected so that they oppose one another magnetically, the total self-inductance is 1 milli-henry.

Find the coefficient of self-inductance of each coil, and that of mutual inductance. (See Section 26 of this chapter.)

Ans. 2·265 and 0·735 milli-henries, $M = 1$ milli-henry.

24. Show how a ballistic galvanometer may be used to measure flux, and how the constant in such a case may be determined.

A ballistic galvanometer was connected in series with the secondary winding of a standard solenoid and a current of 2 amperes was reversed in the primary winding. The first fling observed on the galvanometer scale was 65 divisions, and the tenth swing on the same side was 12 divisions.

The details of the standard solenoid were:

Turns in primary coil: 500. Length: 40 cm. Diameter: 8 cm.

Turns in secondary coil: 500. Diameter: 7·5 cm.

(The secondary coil is short and is situated inside the primary coaxially at its centre.)

Find the change of flux per scale division, for an undamped swing.

If the resistance of the secondary circuit were halved, what would be the observed fling, assuming no damping effect due to friction?

(Cambridge, A, 1912.)

Ans. 4×10^{-7} webers (40 maxwells), 62 divisions.

CHAPTER IV

FERROMAGNETISM

PART I

ELECTRO-MAGNETS

1. The iron-cored toroid.

The magnetic flux set up in a toroidal coil by a given magneto-motive force can be measured by means of a ballistic galvanometer or flux-meter connected to a separate coil (S, Fig. 86) wound on to the ring. The flux is measured by

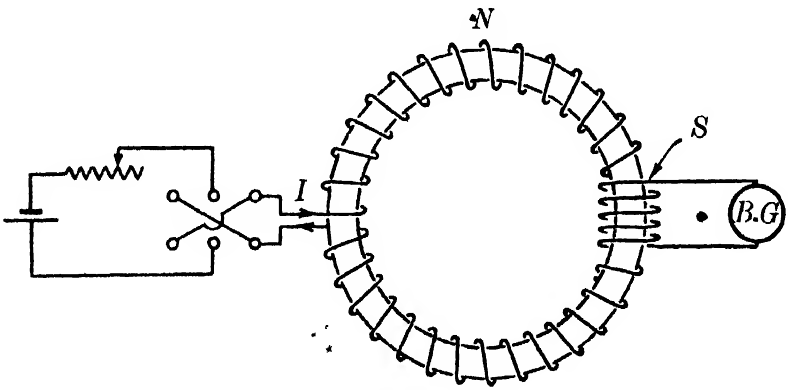


Fig. 86. Iron-cored toroid

recording the deflection of the galvanometer when the exciting current, I, is broken or reversed, and by the use of equation 3(22).

By performing this experiment for different currents (I) on rings of various mean lengths (L) and sectional areas (A), with exciting windings of various numbers of turns (N), we may verify the "law" of the magnetic circuit:

$$m = \phi S, \qquad 3(35)$$

where $$m = IN \quad \text{and} \quad S = \frac{L}{\mu_0 A}$$

and the flux-density

$$B_0 = \frac{\mu_0 IN}{L} = \mu_0 H.$$

Now if instead of winding the coil on a ring of non-magnetic material, we use a ring of iron, we find that the magnetic flux set up by a given current is enormously increased, and we introduce the numerical ratio μ, where

$$\mu = \frac{\text{The flux in the iron ring}}{\text{The flux in the non-magnetic ring}}$$

and is called the *relative permeability* of the iron.

Further, we may put

$$\mu\mu_0 = \mu_a,$$

the *absolute* permeability of the iron.

The "law" of the magnetic circuit (equation 3(35)) still holds if the reluctance, S, is given by

$$S = \frac{L}{\mu\mu_0 A} = \frac{L}{\mu_a A}. \tag{1}$$

The mean flux-density in the iron is then

$$B = \frac{\phi}{A} = \frac{\mu_a IN}{L} = \mu_a H = \mu B_0\,, \tag{2}$$

where $H = IN/L$, the m.m.f. gradient of the *exciting coil.*

We may consider the increase in the flux-density to be due to the *orientation* of the magnetic axes of the iron atoms under the action of the magnetizing field B_0. When the ring is unmagnetized, we imagine the magnetic axes of the atoms to be pointing in random directions (Fig. 87, a) so that they cause no *average* field in any one direction, but upon the application of an external field (Fig. 87, b) the magnetic axes of the atoms turn in an endeavour to align themselves with the magnetizing field, and in so doing contribute an *average* flux-density of value $B_0(\mu-1)$, of which the m.m.f. gradient is of course $H(\mu-1)$.

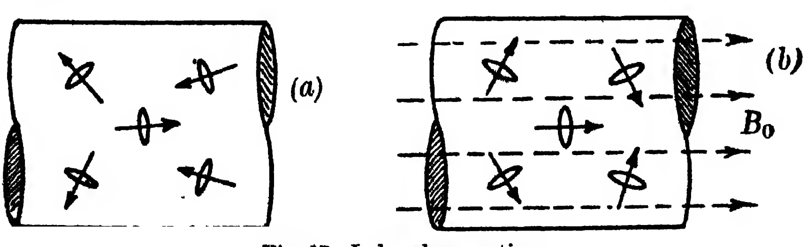


Fig. 87. Induced magnetism

Owing to the relatively enormous distances which apparently exist between the constituents of an atom, we may regard the total magnetic field, of average value B, to exist in a region of free space, in which we occasionally meet electrons and atomic nuclei, and the m.m.f. gradient of this field (the *total* m.m.f. gradient, not to be confused with that of the exciting coil, H) has an average value given by

$$H_t = \mu H \tag{3}$$

and
$$B = \mu_0 H_t = \mu_0(\mu H) = \mu B_0.$$

From this point of view, therefore, μ is a numerical and empirical ratio which enables us to allow for the *averaged* magnetic effect of the atomic currents. The old word "permeability", though retained, is not a very happy one, since it suggests a medium *different* from that of free space, and in which a given total m.m.f. gradient is associated with a different value of B.

In general we may put

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_\iota, \quad \text{vectorially,} \tag{4}$$

where $\mathbf{B} =$ the total average flux-density, $\mathbf{B}_0 =$ the flux-density contributed by the current in the exciting winding, $\mathbf{B}_\iota =$ the flux-density contributed by the atomic currents. For the iron-cored toroid $\mathbf{B} = \mu \mathbf{B}_0$, so that

$$\mathbf{B}_\iota = (\mu - 1)\mathbf{B}_0. \tag{5}$$

In classical theory, based upon the "unit magnetic pole", what we have termed $\mathbf{B}_\iota$ is equal, in this case, to $\mathbf{M}$, where $\mathbf{M}$ is the "intensity of magnetization". The classical form of 4(4), in this particular case, is therefore:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}. \tag{4 a}$$

The ratio M/H is called the magnetic susceptibility, and is usually denoted by κ. Thus

$$\kappa = \frac{M}{H} = \mu_0(\mu - 1). \tag{6}$$

Now the mean flux-density provided by the iron, $(\mu - 1)B_0$, could be provided in the absence of the iron by increasing the m.m.f. of the winding by an amount equal to $(\mu - 1)IN$,

which is therefore the m.m.f. provided by the iron. The gradient of this m.m.f. is uniform around the ring, and is given by

$$\text{m.m.f. gradient of the iron} = H_i = (\mu - 1)\frac{IN}{L}$$

$$= \frac{(\mu - 1)}{\mu\mu_0} B. \qquad (7)$$

Equation 4(7) has been put into this particular form on account of the tacit assumption, in all practical calculations of the magnetic fields in machines, that the m.m.f. provided by unit length of the iron is a *function of* B, the *total* flux-density. That μ is itself a function of B does not affect this result, which may be expressed as

$$H_i = f(B). \qquad (8)$$

2. Magnetization curves.

If the flux-density in the iron ring is measured for increasing values of H (suitable precautions being taken that the specimen is demagnetized before every reading of the galvanometer) the familiar "magnetization curve" (Fig. 88) is obtained. By dividing the values of B by the corresponding value of $\mu_0 H$, a further curve of the relative permeability, μ, may be plotted,

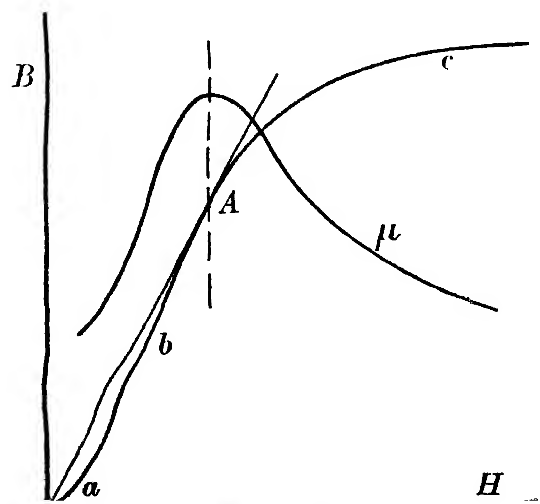


Fig. 88. Magnetization curv

and with the aid of these two curves we may distinguish three main divisions in the phenomena:

(*a*) At the foot of the curve μ rapidly increases and the *B-H* curve bends upwards.

(*b*) Over a certain range of *H* the *B-H* curve is approximately straight, but μ still increases since the straight portion of the magnetization curve does not follow a line which passes through the origin.

(*c*) At the point *A*, at which the tangent to the *B-H* curve passes through the origin, μ is a maximum, and upon increasing *H* we pass into the *region of saturation*, in which μ steadily decreases, and finally would approach unity. The portion above *A* where the curvature is greatest is often called the "knee" of the magnetization curve.

The physical phenomena represented by the curve are extremely complicated, depending upon inter- and sub-atomic forces which are as yet imperfectly understood. The phenomenon of saturation is readily explained by the supposition that the orientation of the atomic magnetic axes is more or less complete, so that the contribution of the atoms to the total field has reached a limiting value, but the initial parts of the curve show evidence of very complex phenomena.

Modern electron theory suggests the "spontaneous" magnetization of perfect ferro-magnetic crystals, which means that, in a perfectly symmetrical arrangement of certain atoms, the stable condition is one of complete orientation of their magnetic axes in one of several "directions of each magnetization". In a piece of unmagnetized iron we may imagine a chaotic arrangement of very small perfect crystals, or "domains", so that their spontaneous magnetization does not result in any aggregate magnetic field. Upon the application of a small magnetizing field, there is perhaps a slight straining of the crystal structure owing to a small rotation of the magnetic axes from the positions of "easy magnetization" in which they happen to be, the result being a moderate contribution to the aggregate field. As *H* increases, there apparently comes a point where the axis of magnetization of a "domain" jumps suddenly from the original direction of "easy magnetization"

to another more nearly aligned with the magnetizing field. A rapid increase in B occurs, and this may be supposed to continue until the magnetic axes of all the domains have been thus changed. Beyond this point, further contributions to B come once more by a slight straining of the domains, and only upon the superposition of an indefinitely great magnetizing field can we imagine that every atomic magnetic axis is perfectly aligned.

Now the majority of atoms possess magnetic moments, but why the usual arrangement of atoms in a substance is such that the magnetic moments cancel out in the aggregate is a question that we cannot discuss here. Apparently the atoms of iron, nickle, and cobalt, and certain combinations of atoms in magnetic alloys, are exempt from this rule, to the great benefit of the engineer. The problems of the magnetic properties of matter are of great importance to the physicist, but the practising engineer is usually content with the empirical information provided by the experimentally determined B-H curve. He is, nevertheless, vitally interested in the development and discovery of more effective magnetic materials by the research worker.*

Typical magnetization curves for commercial sheet steel, cast steel, and cast iron are shown in Fig. 89. They are plotted in three different unit-systems:

(1) Rationalized m.k.s.:	B	in webers per square metre
	H	in ampere-turns per metre
(2) Classical c.g.s.:	B	in gauss
	H	in oersteds
(3) "Engineering" units:	B	in kilo-maxwells per square inch
	H	in ampere-turns per inch

In using these curves, it should be realized that the behaviour of a particular specimen may vary considerably from that of the average, both on account of slight differences in composition and heat-treatment, and on account of its previous "magnetic history".

* For a short account of the physical theory of magnetic materials see R. M. Bozorth, "Present Status of Ferro-Magnetic Theory", *Electrical Engineering*, Nov. 1935, p. 1251. For deeper study see E. C. Stoner, *Magnetism and Matter* (Methuen).

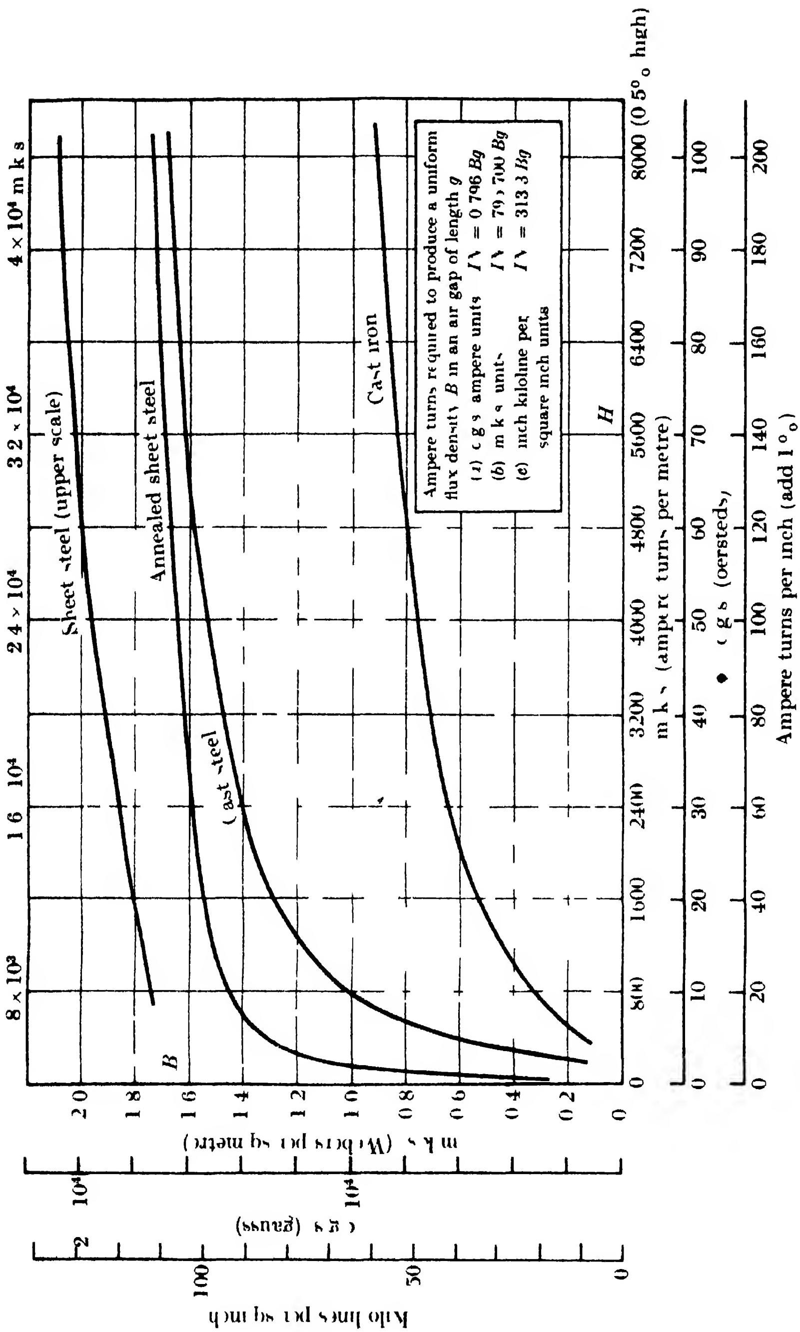


Fig 89 Typical magnetization curves

3. Fröhlich's equation for the *B*-*H* curve.

The non-linear dependence of B upon H is a great stumbling-block to the formulation of accurate theoretical analysis of electrical machines possessing ferro-magnetic parts, and many attempts have been made to represent the B-H curve by a mathematical equation. Perhaps the simplest and most useful equation is that of a rectangular hyperbola, to which many magnetization curves are found to approximate over a limited range. It should be emphasized, however, that the usefulness of such an equation is usually confined to those cases where the magnetic field in the iron is not changing.

Consider the equation

$$B = \frac{aH}{b+H}. \tag{9}$$

This represents a rectangular hyperbola (Fig. 90) passing

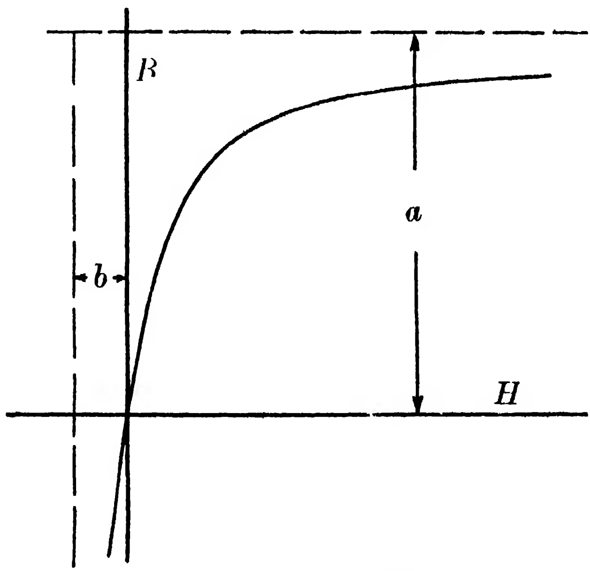


Fig. 90. Rectangular hyperbola

through the origin, with asymptotes $B=a$ and $H=-b$, and for positive values of H has a shape suggestive of that portion of the magnetization curve above the lower portion (*a*), Fig. 88. In order to determine to what accuracy a given B-H

curve can be represented by equation 4(9), we proceed as follows:

From 4(9),
$$b+H = a\frac{H}{B}.$$

That is, if we plot values of H/B upon a base of H we obtain a straight line (Fig. 91) whose intercept on the axis of H is equal to $-b$, and whose intercept on the axis of H/B is equal to b/a (i.e. the slope of the line is equal to $1/a$). Fig. 92 shows values of H/B plotted on a base of H (in unrationalized m.k.s. units) for the three magnetization curves of Fig. 89, and it is seen that the agreement with the form of 4(9) is fairly good for the range considered, except for low values of H. The curve for

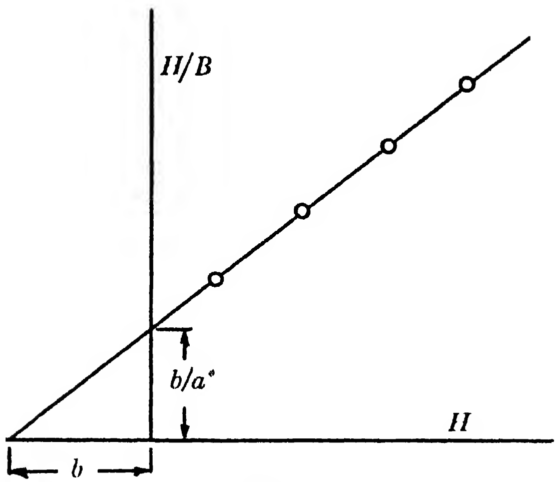


Fig. 91

annealed sheet steel follows the hyperbolic law to quite low values of H. The values of a and b obtained from the straight lines of Fig. 92 give the following equations (in m.k.s. units) for the curves of Fig. 89:

(1) *For cast iron*:
$$B = \frac{1{\cdot}06H}{1610+H} \quad (\text{for } H > 1500).$$

(2) *For cast steel*:
$$B = \frac{1{\cdot}76H}{602+H} \quad (\text{for } H > 700).$$

(3) *For annealed sheet steel:*

$$B = \frac{1{\cdot}66H}{101+H} \quad \text{(for } H > 200\text{)}.$$

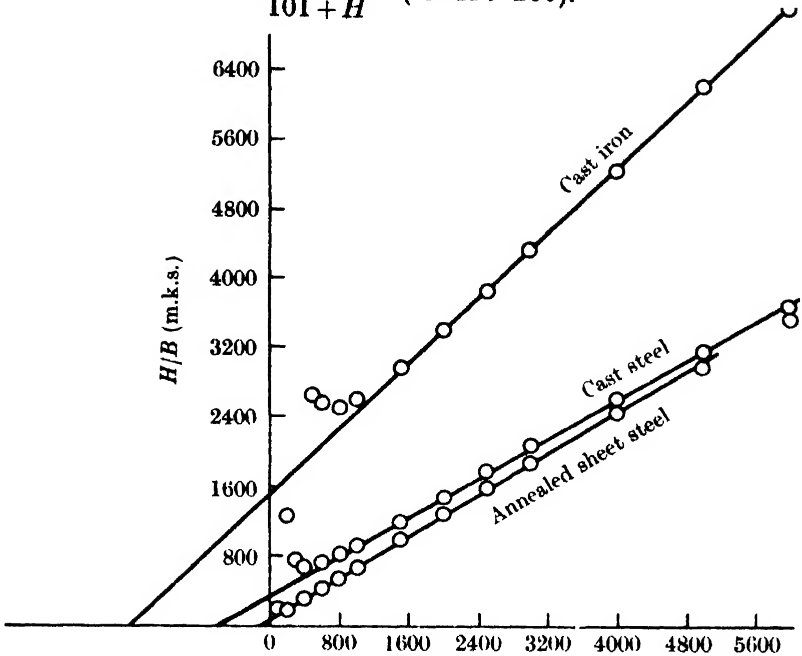


Fig. 92

4. The cycle of magnetization: hysteresis.

When the flux-density, B, in an iron ring (initially in an unmagnetized state) is gradually increased by means of an increasing magnetizing field, the relation between B and H follows the magnetization curve discussed above. Suppose now that a flux-density B_m has been reached, due to an m.m.f. gradient H_m, and the exciting current falls to zero. It will be found that the value of B does *not* fall to zero, but decreases to some value B_r (Fig. 93) called the *residual* or *remanent* flux-density. Evidently the magnetic axes of the atoms retain some degree of orientation when the magnetizing field is removed, probably retaining to some extent the "directions of easy magnetization" to which they jumped under the magnetizing influence of the exciting current. The ring is now a "permanent" magnet, and to reduce B to zero it is necessary

to apply a *reversed* m.m.f., whose gradient $-H_c$ is called the *coercive force* for the particular value of B_m.

An increase of $-H$ to $-H_m$ is found to produce a rapid change of B from zero to $-B_m$, probably due to the rapid change in the directions of spontaneous magnetization of the "domains". A further reversal of H from $-H_m$ to $+H_m$ is accompanied by a similar "lagging" of B, the relation between H and B now being given by the path A'–R'–C'–A in Fig. 93.*

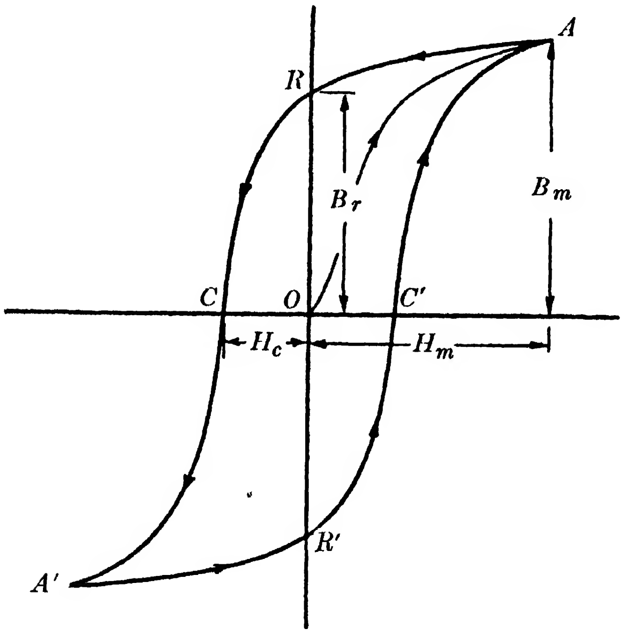


Fig. 93. Hysteresis loop

The phenomenon is known as magnetic "hysteresis", the closed curve of Fig. 93 being called a "hysteresis loop". It follows that, under such circumstances, the value of μ, the relative permeability, depends both upon H and the previous magnetic treatment of the ring, and its value may vary between wide limits. Accordingly, it is usual to define it by the ratio B/B_0 given by the simple magnetization curve, O–A.

* It is found in practice that the first magnetic cycle usually does not result in a completely closed "loop", it being necessary to repeat the cycle several times before this condition is attained.

5. Hysteresis loss.

In taking the ring-specimen of Fig. 86 through a magnetic cycle, let the exciting current in the winding increase by δI in time δt, and let $\delta\phi$ be the corresponding increase in the flux in the ring. Let A and L be the cross-sectional area and mean length of the ring. Then an e.m.f. is induced in the exciting winding due to the change of flux:

$$e = -N\frac{\delta\phi}{\delta t} = -NA\frac{\delta B}{\delta t},$$

so that, to maintain the current, energy must be expended of amount $$\delta W = (-e)\,I\,\delta t = INA\,\delta B,$$

where I is the mean value of the exciting current over the interval δt.

But $IN = HL$, so that

$$\delta W = (LA)\,H\,\delta B,$$

and the total energy, which must be expended during a complete magnetic cycle, is

$$W = \oint dW = (LA)\oint H\,dB \qquad (10)$$

= (Volume of iron) × (Area of hysteresis loop).

This energy is converted into heat, and results in an increase of temperature of the iron. It is called the "hysteresis loss".

The Steinmetz equation. C. P. Steinmetz found that, for a given specimen, the area of the hysteresis loop can be expressed empirically by the relation:

$$\text{Area} = \eta(B_m)^x,$$

where η = the "hysteresis coefficient", depending on the material, B_m = the numerical value of the maximum flux-density, x = approximately 1·6 for ordinary magnetic materials.

We may therefore write, for the hysteresis loss in a symmetrical cycle,

$$W_h = \eta\,.\,B_m^{1\cdot6} \text{ per unit volume, per cycle} \qquad (11)$$

The coefficient η is usually given in c.g.s. units, i.e. in ergs per

cubic centimetre, for values of B_m in gauss. We may convert to m.k.s. units as follows:

Let η_c = the hysteresis coefficient in c.g.s. units.
Let η_m = the hysteresis coefficient in m.k.s. units.
Then

$$W_h = \eta_c(B_m \,.\, 10^4)^{1\cdot 6} \text{ ergs per cu. cm.} \quad (B_m \text{ in w.p.s.m.})$$
$$= \eta_c \,.\, 10^{-7} \,.\, 10^6 \,.\, (10^4)^{1\cdot 6} \,.\, (B_m)^{1\cdot 6} \text{ joules per cu. metre,}$$
$$= \eta_c \,.\, 10^5 \times 2\cdot 512 \times (B_m)^{1\cdot 6} = \eta_m \,.\, B_m^{1\cdot 6},$$

whence

$$\eta_m = 2\cdot 512 \times 10^5 \,.\, \eta_c. \qquad (12)$$

For a typical sample of annealed sheet steel,

$$\eta_c = 10^{-3}, \text{ in c.g.s. units,}$$

so that $$\eta_m = 251, \text{ in m.k.s. units.}$$

For displaced (unsymmetrical) loops, in which the positive and negative values of the maximum flux-density, B_m, are not numerically equal, the Steinmetz equation does not apply, and no satisfactory empirical rule appears to have been found to fit this rather complex case.

6. The iron ring with a concentrated exciting winding.

If an iron ring is wound with a concentrated coil (Fig. 94) which covers only a portion of the ring, we find by experiment that the magnetic flux in the ring is approximately the same as in the case of a uniformly distributed winding with the

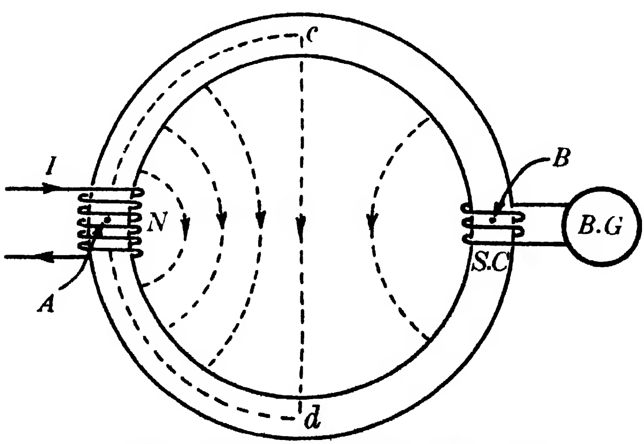


Fig. 94. Iron ring with concentrated winding

same ampere-turns. If we take ballistic galvanometer, or flux-meter, readings for different positions of the search coil ($S.C.$) we find that the flux varies slightly around the ring, being a maximum at the centre (A) of the exciting coil, and a minimum at a point (B) diametrically opposite. The variation is, however, small, and may be of the order of 1 %.

The distribution of the field is thus entirely different from that which would exist if the iron were removed. We may imagine that the orientation of atomic magnetic axes, starting inside the coil, is handed on from atom to atom until the final orientation is very similar to that produced by a uniformly wound ring. Since magnetic flux is a circuital quantity, it follows that a small proportion of the total flux must pass across the air-space. This is called the "leakage flux", and cannot be accurately calculated. A somewhat analogous electrical case is that of a battery inserted in a heavy copper ring, the whole circuit then being immersed in sea water. Since the water is not a perfect insulator, some of the total current from the battery will flow through the water, but this will be a small proportion of that flowing through the copper ring. Again, the distribution of current cannot be accurately calculated.

We may, however, obtain a rough idea of the value of the leakage flux-density along a diameter (c–d) of the ring in the following way. Apply the circuital law to the path (A–c–d–A), and neglect the variation of B within the iron. Let B_L be the average flux-density along the leakage path c–d. Then

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 IN.$$

or $$\frac{B}{\mu}\pi R + B_L 2R \doteqdot \mu_0 IN,$$

where R is the mean radius of the ring. Also

$$\frac{B}{\mu} 2\pi R \doteqdot \mu_0 IN,$$

whence $$B_L \doteqdot \frac{\pi B}{2\mu}.$$

Thus if $\mu = 10{,}000$ B_L is only about 0·016 % of B.

The fact that the flux produced in an iron "circuit" by a concentrated m.m.f. is approximately equal to that due to a uniformly distributed m.m.f. is of enormous practical importance, for it enables us to obtain approximate solutions of problems of which a rigorous theoretical treatment is impossible.

7. Iron ring with a short air-gap.

Suppose the ring of Fig. 94 now has a short uniform air-gap, of length g, cut in it (Fig. 95): what flux will be produced by an m.m.f. of $4\pi IN$?

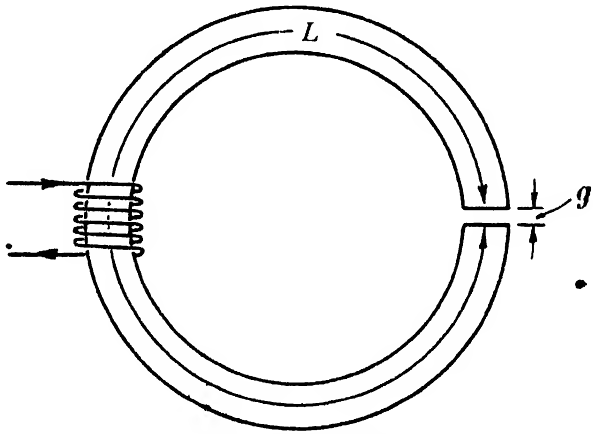


Fig 95. Iron ring with air gap

As a first approximation, we may assume that the configuration of the field is the same as that in a solid ring, and that the flux-density, B, is the same in the gap as in the iron. Assuming that equation 4(7) gives the m.m.f. contributed by unit length of the iron, whenever the total flux-density is B, the *total* m.m.f. acting around the ring is

$$m_t = IN + \frac{\mu - 1}{\mu\mu_0} BL,$$

where L is the mean length of the iron.

The *total* m.m.f. gradient is then

$$H_t = \frac{m_t}{L + g},$$

and the flux-density

$$B = \mu_0 H_i = \frac{\mu_0 IN}{L+g} + \frac{\mu-1}{\mu}\frac{BL}{L+g},$$

whence
$$B = \frac{\mu_0 IN}{g + L/u}. \tag{13}$$

Now in many cases L/μ is small compared with g, so that

$$B \doteq \frac{\mu_0 IN}{g}, \tag{13a}$$

which is the flux-density caused in an air-cored toroid of mean length g by the same m.m.f. ($4\pi IN$). Thus, if the purpose of the coil is to cause a stipulated flux-density B over a region of air of length g, the presence of the iron has the effect of in creasing the available winding space for the coil, and hence of making possible a far higher value of useful flux-density for a given m.m.f.

As an example, suppose that $\mu = 2000$ and $g = L/20$, then L/μ is only 1 % of g, or only 1 % of the m.m.f. is used in the iron, 99 % being absorbed in the air-gap. The distribution of the coil m.m.f. around the circuit is thus entirely changed by the presence of the air-gap, just as the distribution of e.m.f. around a copper loop, connected to a battery, is entirely changed by cutting a gap in the loop and replacing the copper so removed by a metal of high resistivity.

If we assume a constant value of μ, the relative values of flux-density produced by a given toroid, with constant m.m.f., for the three types of "core" considered, are

Air-core	Iron-core with gap	Closed iron core
B_0	$\frac{\mu L}{L+\mu g} B_0 \doteq \frac{L}{g} B_0$	μB_0
If $\mu = 2000$, and $L/g = 20$:		
B_0	$\cdot 20B_0$ (approx.)	$2000B_0$

The m.m.f. required for a given B.

We may put 4(13) in the form:

$$IN = \left(g + \frac{L}{\mu}\right)\frac{B}{\mu_0}$$

$$= \frac{B}{\mu_0} g + HL, \tag{14}$$

where H is the m.m.f. gradient necessary to produce a flux-density B in a closed iron ring. That is, B and H are connected by the usual magnetization curve of a ring specimen.

8. The case of a short iron bar.

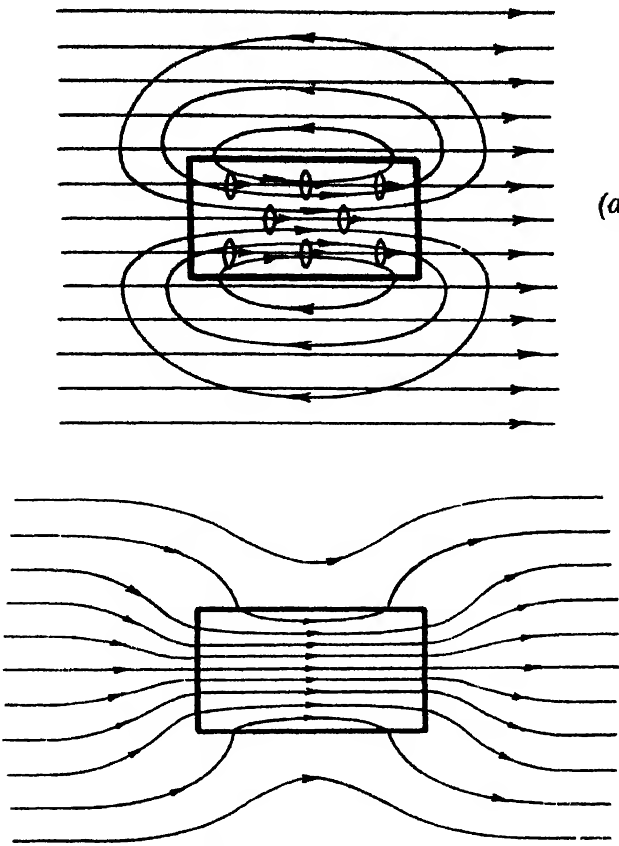
Suppose a small block of iron is placed in a region of free space (or air) occupied by a uniform magnetic field, B_0. Remembering that the apparently solid block is really only a region of space in which there is an extremely attenuated distribution of matter (electrons and atomic nuclei) we have no difficulty in realizing that a magnetic field exists inside the iron. We assume now that the magnetic axes of the atoms turn in such a way as to raise the mean flux-density to some value B. There must, of course, be violent changes in B over distances of atomic dimensions, but we measure the *mean* value over a finite section of the iron. Fig. 96 (*a*) gives a rough idea of the configuration of the two components of the total field: the magnetizing field B_0 and the "induced" field of the iron. These fields combine to form the field distribution of Fig. 96 (*b*), and the flux-density in the iron is again far less than that produced in a closed ring by the same magnetizing field, B_0. In this case we cannot apply equation 4(7), for we cannot guess, even approximately, the relative values of B around any closed path of the total field. We can, however, see two reasons why the field in the iron must be comparatively small:

(*a*) The m.m.f. provided by the iron, for a given B, is proportional to the length of the bar. By analogy with a short solenoid, the thicker the bar the smaller will be the field in its interior for a given m.m.f.

(*b*) The induced m.m.f. of the iron is itself a function of the total induced flux-density [equation 4(7)], so a small field will be attended by a correspondingly small m.m.f.

9. The case of an infinite iron sheet, in an infinite magnetic field.

As a limiting case of the last section, consider a very large thin sheet of iron placed at right angles to the earth's magnetic field (Fig. 97). Then, although the magnetic axes of atoms may



(a)

(b)

Fig. 96. Iron bar in magnetic field

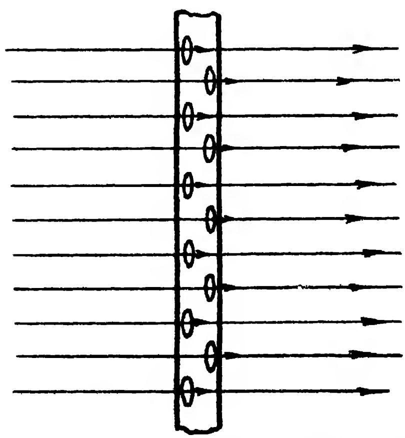


Fig. 97. Thin iron sheet in magnetic field

still be oriented, their m.m.f., compared with the m.m.f. *causing the earth's field, will* be very small and may be neglected, so that no observable change of field will take place. Further, no matter how the inter-atomic field may vary, the average flux-density in the iron must be the same as that in the same space before the iron was introduced, and will be equal to that in the air in the neighbourhood of the sheet.

We conclude, then, that the relation

$$B = \mu B_0 = \mu\mu_0 H,$$

where B_0 and H refer to the conditions when the iron is absent, is true only for certain special cases, such as that of a uniformly wound closed ring, and our conceptions of B_0 and H, in the general case, need clarifying.*

10. The definition of H, the m.m.f. gradient of the magnetizing winding.

Let us denote the m.m.f. gradient provided by the coil at a point, when the iron is absent, by H_0. Then when iron is present, in the general case B and H_0 are not connected by the B–H curve of a ring specimen, for the m.m.f. provided by the iron depends in a most complex manner upon the iron configuration. We are able to reduce the problem of electromagnets to a certain degree of tractability only by making the following arbitrary assumptions.

A. The iron itself is assumed to be without retentivity, and to provide an m.m.f. gradient, given by 4(7), at *points inside the iron only*. At all points exterior to the iron the m.m.f. gradient is assumed to be provided entirely by the magnetizing coil.

B. Following (A), and due to the circuital property of magnetic flux, it is necessary to assume that H, the m.m.f.

* The reader will recognize, in the cases discussed, the "demagnetizing" effects of the induced "surface polarity" of classical theory. For an ingenious discussion, on classical lines, of the iron ring with a concentrated winding, see E. B. Moullin, *Principles of Electro-Magnetism*, p. 164 (Oxford).

gradient provided by the coil at a point, is not the same as H_0, but we must retain the relation

$$\oint \mathbf{H} \cdot d\mathbf{l} = \oint \mathbf{H}_0 \cdot d\mathbf{l} = IN.$$

If equation 4(7) gives the m.m.f. gradient contributed by the iron whenever the flux-density is B, then it follows that the coil must contribute the remainder. The total m.m.f. gradient of the field B is, of course,

$$H_t = \frac{B}{\mu_0},$$

so that contributed by the coil is

$$H = H_t - H_i = \frac{B}{\mu_0}\left(1 - \frac{\mu - 1}{\mu}\right) = \frac{B}{\mu\mu_0}. \qquad (15)$$

Equation (15) provides a very convenient definition of H, for it gives the connection between B and H as given by a test on a uniformly wound ring specimen. Further, magnetic-circuit calculations based on this definition are sufficiently consistent with experiment to justify our acceptance of the fundamental assumption that equation 4(8) is always true. We therefore define H as follows:

DEFINITION OF H IN IRON

> The m.m.f. gradient of a coil, H, at any point in iron where the flux-density is B, is equal to that value of H which corresponds to B in a test on a uniformly wound ring specimen, provided that the magnetic history of the iron is similar to that of the ring specimen.

From this definition it follows immediately that, if we know the value of B at all points of any closed path of a given "magnetic circuit", and if we possess the magnetization curve of the iron, then we may calculate, neglecting the effect of hysteresis, the ampere-turns required in the magnetizing winding. (We have already given a simple example in equation 4(14), for the case of a ring with an air-gap.) Unfortunately, it is usually impossible to determine the *exact* configuration of the field in any given case, but it is not difficult to obtain approximate solutions in the majority of cases met by the

electrical engineer. In obtaining such an idea of a field configuration, the results of Section 11 will be useful.

The meaning of B_0 and B_i in the general case. The m.m.f. gradient of a coil, H, as defined above for a point in a magnetic substance, is in general less than the value of H_0, the m.m.f. gradient at the same point when the magnetic substance is removed. In classical theory, $H = H_0 - H_d$, where H_d is the demagnetizing magnetic force of the induced surface polarity, and obeys the inverse-square law, so that $\oint \mathbf{H}_d \cdot \mathbf{dl} = 0$, while $\oint \mathbf{H}_0 \cdot \mathbf{dl} = IN$, the magneto-motive force of the coil. From the charge viewpoint, however, magnetic poles are pure fiction, and H is merely a quantity which obeys the circuital law

$$\oint \mathbf{H} \cdot \mathbf{dl} = \text{m.m.f.}$$

In classical theory, $B = \mu_0 H + M$, and only in special cases, such as that of a uniformly wound iron ring, is this equivalent to $B = B_0 + B_i$, where $B_0 = \mu_0 H_0$ is the field contributed by the winding, and B_i is the field contributed by the magnetic atoms. In the general case, by the principle of superposition, the presence of the magnetic material will not affect the value of the field contributed by the magnetizing winding, and we shall still use the symbol B_0 to denote this field. *B_0, then, at a point in a magnetic substance, we define as being equal to the flux-density due to external sources.*

In general, then, vectorially,

$$\mu_0 H = \mu_0 (H_0 - H_d) = B_0 - \mu_0 H_d \leqslant B_0,$$

while $$M = B - \mu_0 H = B_i + \mu_0 H_d \geqslant B_i.$$

It is convenient to maintain a symbol, B_0, to denote the component of B due to external sources, for in the theory of electro-magnetic induction in moving bodies, we have seen in Chapter III, Section 2, that, in the most general case, we must distinguish between the components B_0 and B_i.

It would appear, from the charge viewpoint, that B_i has more claim to be regarded as the true "intensity of magnetization" than M, for B_i is the actual flux-density contributed by

the magnetized substance, while M is a quantity derived from the assumption that the flux-density is actually a physical condition in a physical medium, caused by a *magnetic force*, $H = H_0 - H_d$.

By the same reasoning, it appears that the true meaning of μ, the relative permeability, is given by the relation

$$B = B_0 + B_i = \mu B_0,$$

rather than by
$$\mu = \frac{B}{\mu_0 H},$$

for the former equation recognizes the application of the principle of superposition to the magnetic fields of *current circuits* (including atoms), instead of *free magnetic poles*. Many difficulties arise in such a definition of μ, however, for it would then become a function of the configuration of the magnetic material, and would also be a complex quantity, for even in isotropic substances B_0 and B_i would not be in the same direction at a point.

At the present stage in the development of electrical theory, we can do no more than mention these consequences of substituting the charge viewpoint for the classical magnetic-pole method. The artificiality which is thereby introduced into the accepted meaning of permeability is not an obstacle of great practical significance, since we are not usually concerned with numerical values of μ.

11. Conditions at the boundary surface between media of different permeabilities.

As in the analogous case of the boundary surface between two dielectrics (Chapter I, Section 5, p. 25) we shall obtain the conditions for the tangential and normal components of B separately.

(A) *The relation between the tangential components of B.* Let A–B (Fig. 98 a) represent the boundary surface of the two media, of relative permeability μ_1 and μ_2, and let B_{t1} and B_{t2} be the tangential components of the (unchanging) flux-density, near this surface, in the two media, and let H_{t1} and H_{t2} be the corresponding m.m.f. gradients.

Apply the circuital law to the long and thin rectangular path a–b–c–d, whose sides a–b and c–d are parallel to the surface and very close to it, the side a–b being in the medium μ_1, and the side c–d being in the medium μ_2.

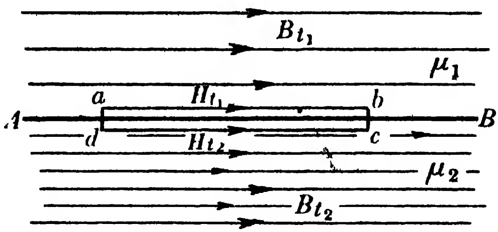


Fig. 98*a*. Boundary surface of magnetic media

Then, since the path is not linked by an electric current,

$$\oint H\,dl = 0,$$

whence it follows that

$$H_{t1} - H_{t2} = 0, \quad \text{or} \quad H_{t1} = H_{t2}.$$

Thus the tangential components of m m.f. gradient are equal, so that

$$\frac{B_{t1}}{\mu_1} = \frac{B_{t2}}{\mu_2}. \tag{16}$$

(Notice that the uniformly wound iron ring is an example of this case.)

(B) *The relation between the normal components.* Let B_{n1} and B_{n2} (Fig. 98*b*) be the normal components of the flux-density near the boundary surface, in the two media. Then, owing to the circuital property of magnetic flux, the flux over any given area of the surface must be the same on either side of the surface. That is,

$$B_{n1} = B_{n2}. \tag{17}$$

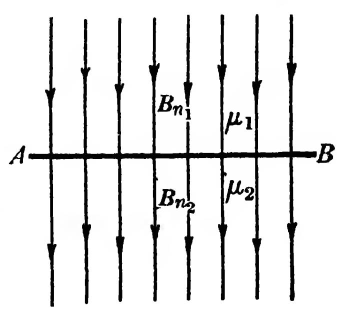


Fig. 98*b*

(Notice that the thin iron sheet, placed normal to the field, is an example of this case.)

(C) *The general case*. Combining the results 4(16) and 4(17), we obtain, from Fig. 98 *c*,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{B_{t1}B_{n2}}{B_{t2}B_{n1}} = \frac{\mu_1}{\mu_2}. \tag{18}$$

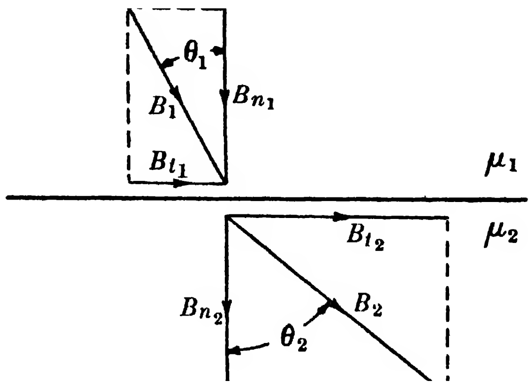


Fig. 98*c*

A common case is that of a boundary between air and iron, for which $\mu_1 = 1$, and $\mu_2 (= \mu)$ is a large number, possibly many thousand. By combining the above results we obtain (see Fig. 99)

$$\begin{aligned} B_{\text{iron}} &= B_{\text{air}}\sqrt{1+(\mu^2-1)\sin^2\theta_1} \\ &\doteqdot B_{\text{air}}\sqrt{1+\mu^2\sin^2\theta_1}. \end{aligned} \tag{19}$$

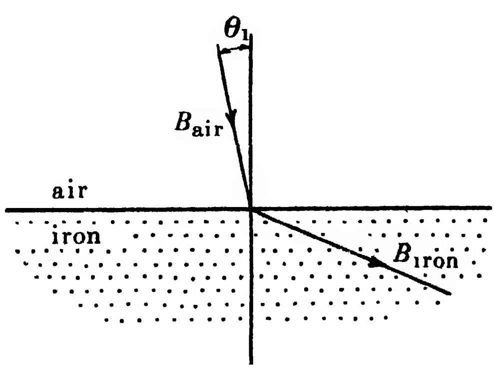


Fig. 99. Boundary surface

Now in an electro-magnetic machine the flux-density in the air-gaps must usually have a specified value, and the usefulness of the iron is lost if B_{iron} is allowed to become excessive. It thus follows from 4(19), since μ is a large number, that θ_1 must in general be a very small angle, and approximate configurations of the magnetic field of a machine may be sketched by assuming that the field enters the iron surface *normally*. That is, $\theta_1 = 0$, which corresponds to infinite permeability, so that the tangential components of H must be taken as zero; this is justifiable only if the iron does not contain embedded conductors carrying current. For example, the configuration of the field between a salient pole and a slotted armature is roughly sketched in Fig. 100.

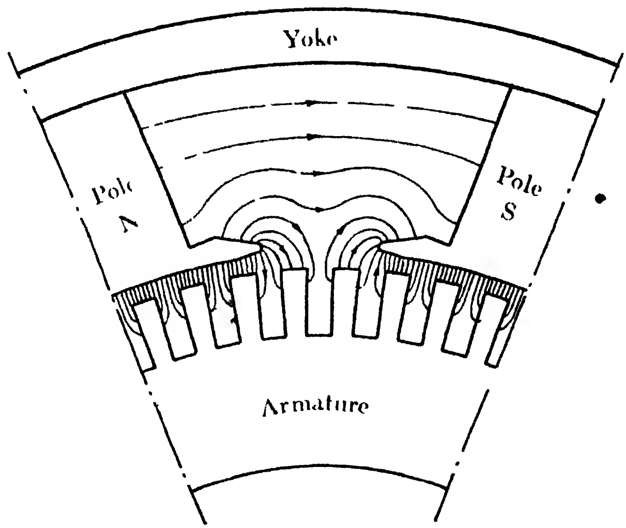


Fig. 100. Magnetic field of salient-pole machine

12. The calculation of magneto-motive force.

If we wish to find what m.m.f. (or ampere-turns) a magnetizing winding must provide, in order to maintain a given flux-density in a given space, we must first determine the complete distribution of the flux. Except in such special cases as that of the iron-cored toroid, an accurate determination of the field distribution is impossible, and the possibility of

as that of the iron-cored toroid, an accurate determination of the field distribution is impossible, and the possibility of approximate calculations of m.m.f. is due to the high relative permeability of iron and the relatively short air-gaps which exist in practical magnet systems.

Suppose that a given flux-circuit can be split up into sections, over each of which the flux-density is sensibly uniform. Let B_1, H_1 and L_1 denote respectively the flux-density, the m.m.f. gradient on a ring specimen to produce this density, and the length, of one section, and let (B_2, H_2, L_2), (B_3, H_3, L_3), etc., denote the same quantities for the other sections. Then the m.m.f. of the winding must be given by

$$m = IN = \oint \mathbf{H} \cdot \mathbf{dl} = \Sigma H_1 L_1. \tag{20}$$

Corrections for leakage and fringing. If the total length of air-path in the circuit is small compared with the total length of the iron, and if any one gap is short in comparison with the linear dimensions of the pole faces, then as a first approximation we may assume that the flux is confined to a path which consists of the iron itself and that part of the air which is enclosed between the opposing iron faces of the air-gaps. Actually, however, there is always some leakage flux and a spreading of the field at the air-gaps, an effect known as "fringing". It is not possible, in general, to subject these effects to accurate analysis, and they are usually allowed for approximately by means of empirical coefficients.*

1. *The leakage coefficient.* Suppose that the purpose of the electro-magnet of Fig. 101 is to provide a flux ϕ_g in the air-gap. Then the total flux, ϕ, set up by the winding is greater than ϕ_g by the amount of the leakage flux. The ratio

$$k_L = \frac{\text{Total flux, } \phi}{\text{Useful flux, } \phi_g}$$

is called the "leakage coefficient", and its value depends upon the shape of the circuit, the position and shape of the winding, and the flux-density in the iron.

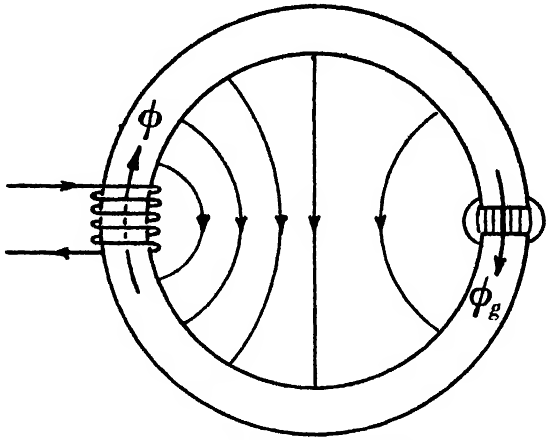


Fig 101 Leakage coefficient

If we wish to reduce the amount of leakage flux, we may learn something from the only known case in which this is practically zero, namely that of a uniformly wound closed iron circuit of uniform section.

Suppose that a leakage flux *docs* exist along any path (shown dotted in Fig. 102) between the two points a and b in

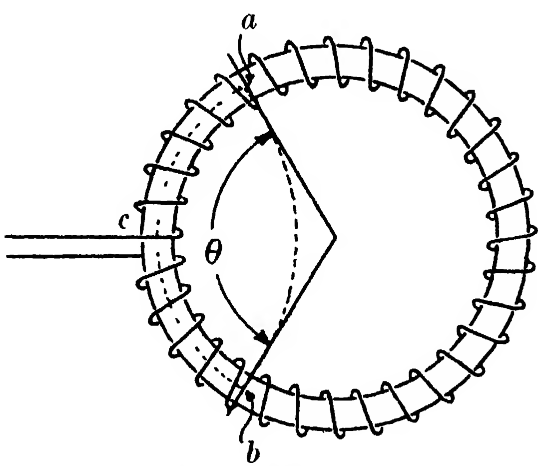


Fig. 102. Leakage flux

a uniformly wound iron ring, and suppose that the mean flux-density along this path is B_L. Then for this flux to exist, there must exist a magneto-motive force between the points a and b, given by

$$m_{ab} = \frac{B_L L}{\mu_0},$$

where L is the length of the path. The circuital law gives, for the ring,

$$IN = \frac{2\pi RB}{\mu\mu_0},$$

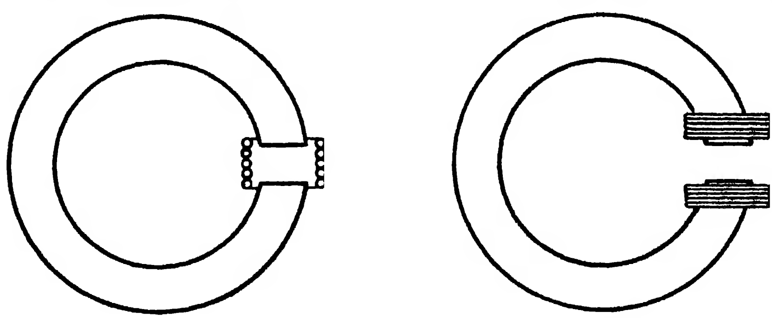
where B is the mean flux-density in the iron, and for the assumed path a-b-c:

$$IN\frac{\theta}{2\pi} = \frac{\theta RB}{\mu\mu_0} + \frac{B_L L}{\mu_0},$$

whence it follows immediately that $B_L = 0$.

Stating the reason in words, we say that there can be no net m.m.f. between the points a and b because the m.m.f. available around the path a–b–c is entirely absorbed in the field in the iron, along the path a–c–b.

In order to reduce leakage, therefore, we should apply ampere-turns, as far as possible, where they are needed. In the case of a ring with an air-gap practically all the available m.m.f. is used in the gap, so that the coil should theoretically be arranged as in Fig. 103, a. However, this arrangement is scarcely practical if we wish to have free access to the field in the gap, and the next best method is to split the winding into two short coils, placed one on each side of the gap and as close to it as possible (Fig. 103, b). A uniformly distributed winding



(a) (b)

Fig 103

will not prevent leakage in this case, since the m.m.f. gradient provided by the coil is not uniformly distributed.

2. *Fringing.* The flux, as it passes across the gap of the incomplete ring, is not confined to the same area as that of the pole face, but bulges outwards as shown in Fig. 104. A little thought will show that the same m.m.f. is available, in passing across the gap, whether we follow a straight line of force in the centre of the pole face or one of the curved "fringing" lines at the edges. Consequently, the mean flux-density along a curved line of length f will be given by

$$B_f = \frac{g}{f}B,$$

where B is the density at the centre of the pole face.

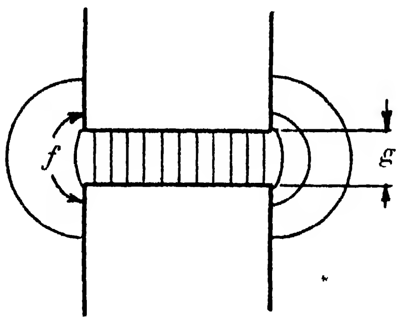


Fig. 104. Fringing

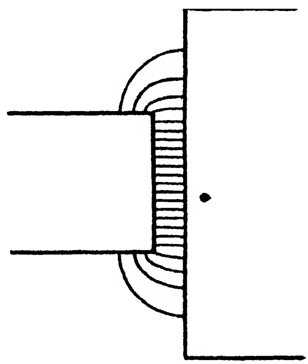


Fig. 105. Fringing

Since the flux in the gap is spread over a larger area than in the iron, the effect of fringing is to decrease the mean flux-density in the gap. An empirical correction may be made by assuming that the flux is uniformly distributed over an "effective" area obtained by adding a strip of width $\frac{1}{2}g$ around the actual area of the pole face. For example, suppose the pole face is circular and of radius r. Then the effective area of the gap is taken to be

$$A' = \pi(r + \tfrac{1}{2}g)^2.$$

If one pole face is very much larger than the other (Fig. 105), a rough value of the effective area is obtained by adding a strip of width g around the area of the *smaller* face.

Carter's fringing coefficient. An important case of fringing is that of the flux passing between the teeth of a slotted armature and an adjacent pole-piece (Fig. 106). F. W. Carter* has shown that the effect may be allowed for by assuming that the mean flux-density in the gap exists across an effective *uniform* gap of length g' given by

$$\frac{g'}{g} = \frac{p}{t+fs}, \tag{21}$$

where p = the tooth pitch, t = the width of the top of a tooth, s = the width of the top of a slot, and f is a function, less than

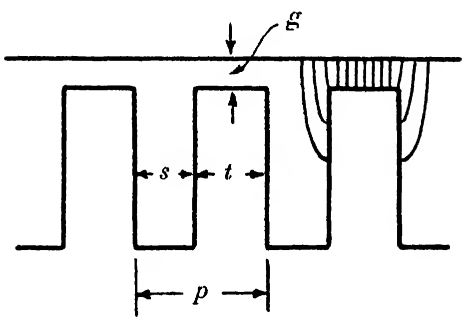


Fig. 106. Fringing at armature teeth

unity, of s/g. The variation of f with s/g is shown in Fig. 107. The ratio g'/g is usually known as the "Carter Coefficient". It is interesting to note that the correction is made by means of an *effective gap length*, instead of an *effective area*.

An example of the calculation of m.m.f., in m.k.s. units. We shall take the case of the magnetic circuit of a 6-pole, 275-volt, 500 kilowatts, 900 r.p.m. direct-current generator. The armature has 108 slots, and the normal flux per pole is about $8{\cdot}7 \times 10^{-2}$ weber (8·7 megalines). The leakage coefficient for this type of machine may be taken to be 1·15. See Fig. 108.

The procedure is as follows:

(1) Determine the flux in each part of the circuit.

(2) Determine the effective areas of each part.

* F. W. Carter, "Note on Air-Gap and Interpolar Induction", *Jl. I.E.E.* XXIX (1900), p. 925; also "Air-Gap Induction", *Electrical World and Engineer*, XXXVIII (1901), p. 884.

(3) Dividing the flux by the area, we obtain the value of B in each part.

(4) From the magnetization curves of the materials we find the m.m.f. gradient, H, required for each part. In the case of the air-gap $H = B/\mu_0$.

(5) The effective lengths of each part are then determined. We wish to find the ampere-turns to be provided on each pole: a complete path of the flux is shown at $(a\text{–}b\text{–}c\text{–}d\text{–}c'\text{–}b'\text{–}a)$ and links with the windings of *two* poles. Hence one pole has to

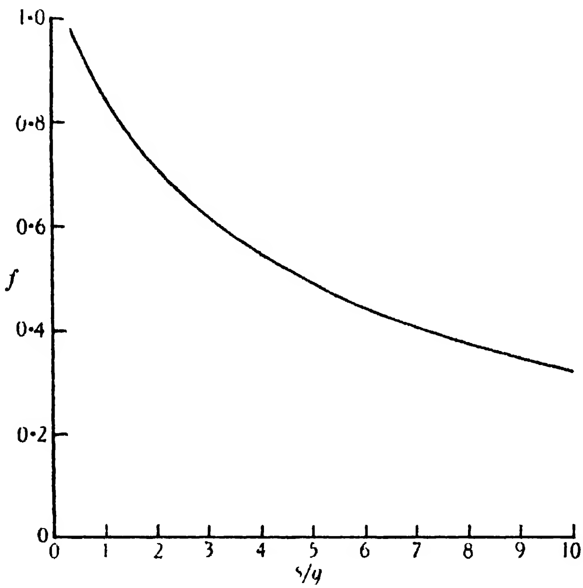


Fig. 107. Carter's fringing coefficient

maintain the flux over half this path, or over the path $(a\text{–}b\text{–}c\text{–}d)$. A similar path is in parallel with $(a\text{–}b\text{–}c\text{–}d)$, so if we calculate the m.m.f. required for the latter, the same flux will be maintained in the parallel path by the same m.m.f.

(6) The product HL for each part gives the m.m.f. required by that part, and the total m.m.f., per pole, is obtained by summing the values of HL [see equation 4(20)].

The dimensions of the machine are given in metres in Fig. 108, and the results of the calculations are tabulated below. Certain of the calculations, however, need some explanation.

Effective area of armature teeth.

The slot pitch $= \dfrac{2\pi \times 0{\cdot}4}{108} = 0{\cdot}0233$ metre,

Width of tooth at top $= 0{\cdot}0233 - 0{\cdot}0091 = 0{\cdot}0142 = t_1$,

Slot pitch at bottom of slots $= \dfrac{2\pi \times 0{\cdot}372}{108} = 0{\cdot}0216$,

Width of tooth at bottom of slot $= 0{\cdot}0216 - 0{\cdot}0091$
$= 0{\cdot}0125 = t_2$.

Owing to the high flux-density in the teeth, the value of H required at the bottom of a tooth is considerably greater than that required at the top, and a mean value of H corresponding

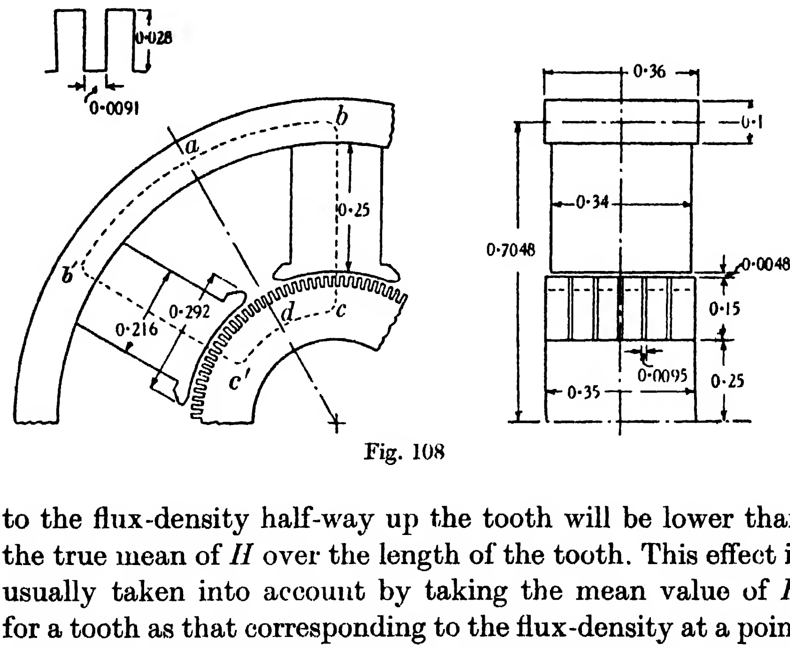


Fig. 108

to the flux-density half-way up the tooth will be lower than the true mean of H over the length of the tooth. This effect is usually taken into account by taking the mean value of H for a tooth as that corresponding to the flux-density at a point one-third of the length of a tooth from the bottom. I.e. the *effective* width of a tooth is taken to be

$$\frac{t_1 + 2t_2}{3} = 0{\cdot}013.$$

The effective length of the armature is the actual thickness of iron. There are five ventilating ducts, and the armature laminations are separated by a layer of insulating varnish or paper, which we may assume accounts for 10 % of the total

thickness of the laminations. The effective axial length of iron is then

$$0{\cdot}9 \times (0{\cdot}35 - 5 \times 0{\cdot}0095) = 0{\cdot}272 \text{ metre.}$$

To obtain the number of teeth carrying the flux ϕ_g, we may make the usual empirical allowance for fringing at the pole tips, by adding a length g at each end of the pole arc. The number of teeth carrying flux is then

$$\frac{(0{\cdot}292 + 2 \times 0{\cdot}0048)}{0{\cdot}0233} = \frac{0{\cdot}3016}{0{\cdot}0233} = 13.$$

Thus the effective area of the teeth carrying flux is

$$13 \times 0{\cdot}272 \times 0{\cdot}013 = 4{\cdot}61 \times 10^{-2} \text{ sq. metre.}$$

Area of the armature core below the teeth:

$$= 0{\cdot}272 \times (0{\cdot}15 - 0{\cdot}028) = 3{\cdot}32 \times 10^{-2}.$$

Length c–d of the armature core:

$$\text{Mean radius} = 0{\cdot}25 + \frac{0{\cdot}15 - 0{\cdot}028}{2} = 0{\cdot}311,$$

$$\text{The length } c\text{–}d = \frac{2\pi \times 0{\cdot}311}{12} = 0{\cdot}1625.$$

Length a–b of the field yoke:

$$= \frac{2\pi \times 0{\cdot}7048}{12} = 0{\cdot}368.$$

Part of circuit and material	Flux	Area	B Flux-density	H (from Fig. 89)	L Length	HL m.m.f.
Field yoke (cast steel)	$\frac{1}{2}(1{\cdot}15\phi_g)$ $5{\cdot}0 \times 10^{-2}$	$3{\cdot}60 \times 10^{-2}$	1·390	$2{\cdot}31 \times 10^3$	a–b 0·368	852
Pole core (cast steel)	$(1{\cdot}15\phi_g)$ $10{\cdot}0 \times 10^{-2}$	$7{\cdot}34 \times 10^{-2}$	1·360	$2{\cdot}25 \times 10^3$	0·25	517
Air-gap	ϕ_g $8{\cdot}7 \times 10^{-2}$	$10{\cdot}25 \times 10^{-2}$ $(0{\cdot}3016 \times 0{\cdot}34)$	0·848	B/μ_0 $6{\cdot}78 \times 10^5$	$5{\cdot}39 \times 10^{-3}$	3660
Armature teeth (sheet steel)	ϕ_g $8{\cdot}7 \times 10^{-2}$	$4{\cdot}61 \times 10^{-2}$	1·885	$1{\cdot}86 \times 10^4$	0·028	520
Armature core (sheet steel)	$\frac{1}{2}\phi_g$ $4{\cdot}35 \times 10^{-2}$	$3{\cdot}32 \times 10^{-2}$	1·310	402	c–d 0·163	66
						$\Sigma HL = 5615$

From 4(20), $IN = \Sigma HL = 5615$ ampere-turns per pole (0·5 % high).

Effective length of air-gap:

$$s/g = \frac{0{\cdot}0091}{0{\cdot}0048} = 1{\cdot}9 \quad \text{(from Fig. 107, } f = 0{\cdot}72\text{)}.$$

$$\text{The Carter Coefficient} = \frac{p}{t_1 + fs} = \frac{0{\cdot}0233}{0{\cdot}0142 + 0{\cdot}72 \times 0{\cdot}0091}$$
$$= 1{\cdot}123,$$

hence the effective length $g' = 1{\cdot}123g = 1{\cdot}123 \times 0{\cdot}0048$
$= 0{\cdot}00539.$

13. To find the flux caused by a given m.m.f.

If it is necessary to find what magnitude of flux is caused by a given number of ampere-turns in the magnet system of a machine, the problem cannot be solved directly since the apportionment of the m.m.f. gradient H is impossible until the flux-densities in all parts of the circuit are known. In other words, we need the values of μ for our calculations, and cannot obtain them until the answer is known. We may proceed in one of three ways:

(*a*) Use the method of trial and error.

(*b*) Apply the best form of equation 4(9) to the magnetization curves.

(*c*) Find the m.m.f.'s required for various values of flux, and draw a magnetization curve for the circuit.

Of these, the third is the method most used in design work.

14. The force between the poles of a magnet.

The opposing "pole faces" of the ring of Fig. 95 attract one another. According to the classical theory of magnetostatics, this attraction is thought to be due to the forces of attraction between the "induced poles" on the faces of the gap, but we prefer to think of it as the force of attraction between atomic current circuits, in the opposing poles, which magnetize in the same direction.

The energy stored in the field in the air-gap is, per unit volume

$$W = \frac{HB}{2},$$

whence it follows, as in Chapter I, Section 10 (p. 35), that the force of attraction between the faces is

$$F = \frac{HB}{2} = \frac{B^2}{2\mu_0} \text{ per unit area of surface.} \qquad (22)$$

The practical utility of equation 4(22) is limited to those cases where B is approximately constant over the area of the opposing faces. A more general expression is given in the next section.

15. The mechanical efficiency of a lifting electro-magnet, with constant exciting current.

The forces of attraction between opposite "poles" of an electro-magnet are used extensively in practice to perform mechanical work, such as that necessary to close a switch. A simple form of such a magnet is shown in Fig. 109, in which the pivoted armature, A, is attracted to the opposing limb, B.

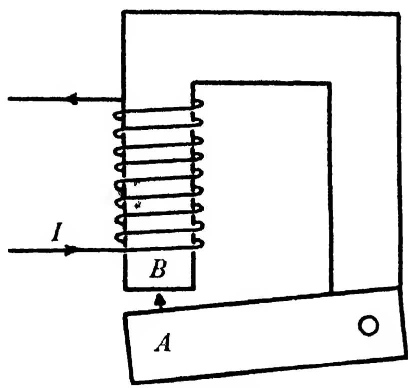


Fig. 109. Electro-magnet

Consider a small movement, δx, of the armature A towards the limb B in time δt. During this movement the total flux in the circuit will increase by some small amount $\delta\phi$, so that to maintain the magnetizing current I at its constant value, additional electrical energy will be required of amount

$$\delta W = (-e) I \delta t, \quad \text{where } e = -N\frac{\delta\phi}{\delta t},$$
$$= IN\delta\phi.$$

Corresponding to the flux increment $\delta\phi$, the inductance of the coil will increase by $\delta L = \dfrac{N\,\delta\phi}{I}$, so that the energy stored in the magnetic field will increase by

$$W_m = \tfrac{1}{2}I^2\delta L = \tfrac{1}{2}IN\,\delta\phi.$$

That is, 50 % of the energy input is stored in the magnetic field, leaving the remaining 50 % available for conversion into mechanical work.

Hence, neglecting the fact that the additional energy stored may be recovered later, the efficiency of the operation, neglecting ohmic losses in the coil, is 50 %.

It follows that the force of attraction between the poles is

$$F = \tfrac{1}{2}IN\frac{d\phi}{dx}. \tag{23}$$

It is readily shown that equation 4(23) reduces to the form of 4(22) in the case of an iron ring with a small gap. In this case $\delta x = -\delta g$, and since $\phi = BA$, the force per unit area of the pole faces is

$$F = -\tfrac{1}{2}IN\frac{dB}{dg}.$$

Now from 4(13), in the gap:

$$B = \frac{\mu_0 IN}{g + L/\mu}$$

$$H = \frac{B}{\mu_0}$$

$$\frac{dB}{dg} = -\frac{\mu_0 IN}{(g + L/\mu)^2},$$

whence $F = BH/2$ per unit area of pole face.

16. The e.m.f. induced in an armature winding in slots.

The armature conductors of dynamo-electric machines are usually located in axial slots in the periphery of a laminated iron rotor or stator. By this means the air-gap between a pole and the armature iron is diminished, so diminishing the exciting m.m.f. necessary to give a specified working flux.

The simplest method of deducing the mean value of the e.m.f. induced in each conductor of a direct-current machine is to apply the flux-cutting law in the form given on page 98. For instance, let

ϕ = the total flux per pole,
n = the speed, in revs. per second,
p = the number of poles.

Then the average e.m.f., per conductor, is equal to the flux cut by that conductor in one second, or

$$E = p\phi n, \qquad (24)$$

a result which is well established as being consistent with experiment.

When, however, we examine the actual flux-distribution in a slotted armature (Fig. 100), we may be disconcerted to find that the conductor whose e.m.f. is so easily calculated does *not* cut through a flux ϕ every time it passes a pole. In reality it lies in a magnetic field whose density is far smaller than the average flux-density over the pole face, for nearly all the flux enters the armature *through the teeth*. Evidently the above method of calculation, although giving the correct result, does not represent the physical facts. Perhaps we may gain a clearer insight into the phenomena by dividing the induced e.m.f. into two parts:

(1) The e.m.f. due to the motion of the conductors in the magnetic field of the moving charges (currents) in
  (*a*) the field winding,
and (*b*) the iron poles.

(2) The e.m.f. due to the rotation of the magnetic axes of the iron atoms in the armature. That is, the direction of the magnetic field of these atoms is continually changing, as viewed from the armature conductors. Consequently an armature coil is linked by a changing field.

The first part is a true flux-cutting e.m.f., but the second is a transformer e.m.f. due to a change of linking flux. Since we may also calculate the e.m.f. (1) by the flux-linking rule,

perhaps the method which is most mentally satisfying is to calculate the whole of the e.m.f. by this rule, as follows:

The e.m.f. at any instant in a single turn (two conductors spaced apart by a pole-pitch, and connected together to form a single loop) is

$$e = -\frac{d\varphi}{dt},$$

where φ is the total flux linking the coil at the instant. The time taken for the coil to travel over one pole-pitch is $1/pn$ seconds, so that the mean value of e, per conductor, is

$$E = \tfrac{1}{2}pn\int_{t=0}^{t=1/pn} e\,dt$$

$$= -\tfrac{1}{2}pn\Big[\varphi\Big]_{t=0}^{t=1/pn}.$$

Choosing $t=0$ when $\varphi = +\phi$ (the flux per pole), when $t=1/pn$ φ will be equal to $-\phi$, for the coil will be opposite a pole of reversed polarity, so that

$$\Big[\varphi\Big]_{t=0}^{t=1/pn} = -2\phi,$$

and the mean e.m.f. per conductor is

$$E = p\phi n$$

as before.

There is, of course, no objection to the use of the very simple flux-cutting rule, for *calculation purposes*, so long as it gives us the correct result. The purpose of the above discussion is to emphasize the fact that the case is not that of a conductor moving through an unchanging field, and is best considered as a combination of both relative-motion and transformer action.

17. The torque developed by a slotted armature.

We have seen in the last section that the e.m.f. generated in an armature winding in slots is the same as though the conductors were actually moving in a field whose density is equal to the mean flux-density in the air-gap. This induction of e.m.f. in conductors which are situated in a very weak field (in the slots) reminds us, in part, of the e.m.f. induced in a

transformer winding, the conductors of which are situated in a field which is negligibly small compared with the field in the iron core.

Now in every dynamo-electric machine on load, and running at constant speed, an electro-magnetic torque is developed which balances exactly the mechanical (driving or driven) torque. Further, the e.m.f. in a conductor moving through a magnetic field is given by

$$E = BLv,$$

while the force on the conductor when it carries a current I is

$$F = BLI.$$

The power developed is

$$P = Fv = BLI\frac{E}{BL} = EI,$$

which is the rate of conversion of energy from one form to another.

It thus follows, that if the e.m.f. per conductor in a slot can be calculated by the relation $E = BLv$, then the force on the armature, per conductor, when carrying a current I can be calculated from the relation $F = BLI$, since

$$\text{force per conductor} = F = \frac{EI}{v} = BLI,$$

where B represents the mean value of the density of the flux which links both armature and field circuits (i.e. the air-gap flux)

Now in an air-cored motor, such as that of a direct-current watt-hour meter, the torque is due entirely to forces on the *conductors themselves*, but when iron is introduced the greater part of the torque is due to forces between the iron surfaces of poles and armature.

Consider the simple case of Fig. 110 (a), which shows two simple coaxial circuits, both carrying current. It has been shown that the mutual force between them is

$$F = I_1 I_2 \frac{dM}{dx}, \qquad 3(46)$$

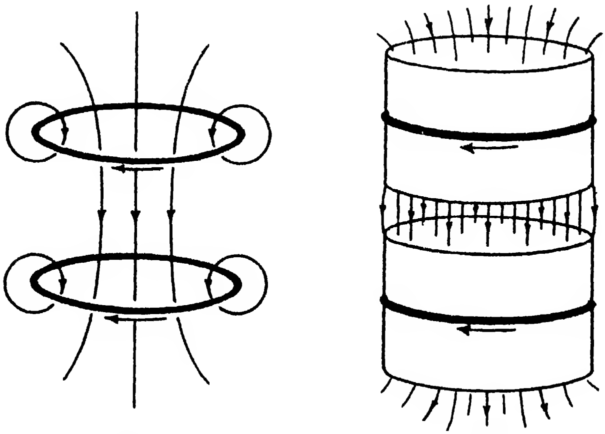
where M is the coefficient of mutual inductance and x is the

distance between them. In this case, it is clear that this force must be borne entirely by the conductors themselves.

Now suppose that the wires are located in grooves in iron cylinders, as in Fig. 110 (*b*), then the reasoning by which we deduced equation 3(46) is unaltered, so that the relation is still valid. The conductors, however, are now situated in a very weak field and the mutual force is borne almost entirely *by the iron.*

In the case of a rotating armature coil, the torque is given by

$$T = I_1 I_2 \frac{dM}{d\theta}$$



(*a*) (*b*)

Fig. 110

and provided that the mean value of $dM/d\theta$ is the same for all cases, it follows that the *total* torque is unaltered by embedding the conductors in slots, although in this case the mutual force will act chiefly upon the iron.

It is often stated that when a magnetic field passes obliquely from one medium to another, with a change of relative permeability, a mechanical force exists on the boundary surface which has a component tangential to the surface. Now if a force on a body has a component in a given direction, then a small motion of the body in the direction of this component must involve work, and hence an energy exchange. This application of the principle of the conservation of energy has

already been used to obtain the results of Sections 14 and 15 above. If now an iron body, assumed homogeneous and with negligible hysteresis, situated in the magnetic field of an external circuit (i.e., the iron is not part of this circuit), moves in such a way that the motion of any point on the surface is everywhere tangential to the surface (e.g., in the rotation of an iron cylinder about its axis), there will be no change in the magnetic energy of the system, and hence there can be no tangential component of mechanical force on an iron surface in the magnetic field of an external circuit. The same result follows from Maxwell's theory of stresses in a magnetic field, as has been pointed out by W. H. Ingram.*

In the usual type of dynamo-electric machine, with salient poles and an armature winding in open slots, the sides of the poles and the armature teeth provide surfaces which are roughly radial to the axis of rotation. Magnetic forces on these surfaces can therefore contribute to the torque of the machine, provided the magnetic field is unsymmetrical with respect to opposite sides of poles or teeth. When no armature current flows, the magnetic field usually has this type of symmetry, and the electro-magnetic torque is then zero. With armature current, however, the symmetry is disturbed by the m.m.f. of this current (i.e. by the armature reaction), and so a torque exists, a small part of which is transmitted by forces on the armature conductors themselves, but the greater portion by forces acting on the sides of the poles and armature teeth.

When conductors are located in closed slots (i.e. tunnels) in iron, as is sometimes the case in the rotors of induction motors, and compensating windings in the pole faces of direct-current machines, the torque is still shared by the conductors and by the iron, the walls of the tunnels again providing iron surfaces which are not everywhere tangential to the direction of motion.

The case of the homopolar generator. An interesting case, which has given rise to a great deal of discussion, is that of action and reaction of mechanical forces in a homopolar generator or motor (as examples, consider Faraday's disc,

* See *The Electrician*, Sept. 13, 1935, p. 311.

Fig. 29, and the homopolar generator, Fig. 42). The essential feature of all such machines is that the conducting circuit, in which the e.m.f. is generated, does not move as a whole, but presents an unchanging configuration, and is situated in an unchanging magnetic field. The e.m.f. is due, not to a motion of the circuit as a whole, or to a changing magnetic field, but to relative motion between the two parts of the circuit (e.g. between the rotating disc and the connecting leads of Fig. 29). Under steady conditions, if no change of magnetic energy is caused by a small rotation of the magnet, then no net force or torque can exist on the iron.

When a rigid coil, carrying a steady current, is situated in an unchanging magnetic field, it will experience a resultant force or torque only if a small lateral or rotary movement results in a change in magnetic energy (or, in other words, a change in the flux linking the circuit). Suppose now that a rigid coil is so situated in a magnetic field that there is no resultant force or torque when a current flows. Individual elements of the coil will experience forces, but these, taken over the complete circuit, will balance. Imagine that a portion of the coil is now cut away and replaced by a disc or cylinder which is capable of rotation, and which makes contact with the two open ends of the remaining portion of the wire. Then, when current flows, the moving and fixed portions of the circuit will experience equal and opposite forces, and the pivoted member will rotate. This does not alter the original fact that no force exists between the circuit and the source of the magnetic field (e.g. the magnet), so that action and reaction exist between the moving and fixed parts of the circuit, and not between magnet and circuit. By the $F = BLI$ law, there will be action and reaction between magnet and disc, and between magnet and stationary leads, but these produce no torque on the magnet since they cancel. If however the magnet forms part of the circuit (e.g. if the arrangement of Fig. 45, p. 114 is used as a motor) the magnet does experience tangential forces.

The reader should examine, in the light of the above discussion, the complete circuit of cylinder and connecting leads

in the homopolar generator of Fig. 42. It is possible to arrange one brush on the cylinder so that its connecting lead does not pass through the magnetic field of the machine, but the lead to the second brush, at the other end of the cylinder, must always pass either through a tunnel in the iron, or through a complete air-gap in the magnetic circuit. In the former case, part of the reaction will be borne by the iron in the vicinity of the tunnel, for a slight rotation of the iron, keeping the lead in the tunnel fixed, will cause a redistribution of the magnetic field set up by the current in this lead, and a change in the magnetic energy. In the latter case, however, there will be no torque tending to turn the iron about the axis of the rotating cylinder.

18. Eddy current loss in iron with alternating flux: core loss.

In most electro-magnetic machines some (or all) of the iron carries an alternating flux. For example, the flux in the rotating armature of a direct-current generator or motor is pulsating, when viewed from the armature itself, and in transformers the whole of the iron core carries an alternating field. Now since iron is an electrical conductor, it follows that currents will be induced in any core which carries a periodically changing field, since this will cause an e.m.f. in the iron. These are called *eddy currents*, and cause an additional loss of energy which results in an increase in the temperature of the iron and a decrease in the efficiency of the machine. This loss may be reduced by building the core of thin sheets or laminations, and by using steel with a high resistivity. The dependence of the loss upon the frequency, maximum flux-density, thickness of laminations, and the resistivity of the steel may be deduced in the following way.

Consider a rectangular block of an iron lamination, of thickness t, length L much greater than t, and unit depth (Fig. 111). The laminations must be parallel to the magnetic field, which we shall assume is perpendicular to the paper and given by

$$B = B_m \sin \omega t.$$

Consider the e.m.f. induced in a closed circuit consisting of two parallel elementary laminae of thickness δx, each distant x from the centre line of the lamination, joined at top and bottom of the length L by short paths of length $2x$.

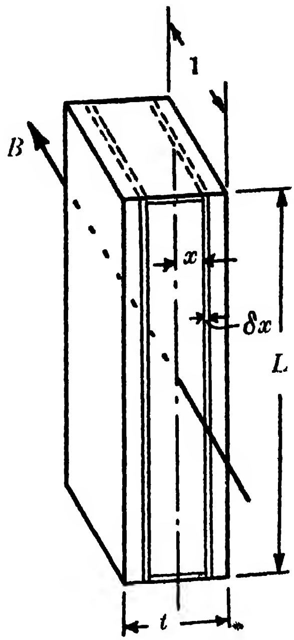


Fig. 111 Eddy-current loss

The flux linking this elementary circuit is

$$\phi = 2LxB_m \sin \omega t,$$

so the induced e.m.f.

$$e = -\frac{d\phi}{dt} = -2Lx\omega B_m \cos \omega t.$$

The resistance of the circuit

$$= R \doteq \frac{\rho 2L}{\delta x}$$

(neglecting the lengths $2x$ in comparison with L), where ρ is the resistivity of the iron, and the ohmic loss

$$= \frac{e^2}{R} = \frac{2Lx^2\omega^2 B_m^2 \cos^2 \omega t\, \delta x}{\rho};$$

the mean rate of this loss, since the mean value of $\cos^2 \omega t = \frac{1}{2}$, is

$$\delta w_e = \frac{L\omega^2 B_m^2 x^2 \delta x}{\rho},$$

hence the mean rate of ohmic loss due to the eddy current, in the complete block, is

$$w_e = \frac{L\omega^2 B_m^2}{\rho}\left[\frac{x^3}{3}\right]_0^{\frac{1}{2}t}$$

$$= \frac{L\omega^2 B_m^2 t^3}{24\rho}.$$

But the volume of the block is Lt, so that the eddy-current loss per unit volume is

$$W_e = \frac{\pi^2 f^2 B_m^2 t^2}{6\rho} \tag{25}$$

where $f = \omega/2\pi$, the frequency of the alternating field in cycles per second,

B_m = the maximum flux-density,

t = the thickness of the laminations,

ρ = the resistivity of the iron.

Equation 4(25) should not be regarded as an accurate expression for eddy-current loss. The postulated current distribution is certainly not that which actually exists, and by Lenz's law the induced currents tend to reduce the flux-density in the iron. This latter effect is appreciable only at super-supply frequencies,* however, and the general form of 4(25), showing that the loss varies as

(*a*) the square of the frequency,
(*b*) the square of the maximum flux-density,
(*c*) the square of the thickness of the laminations,
and (*d*) inversely as the resistivity of the material,
is fairly well established.

For the total *core loss* in an iron core which carries an alternating flux of frequency f, we may write

$$W_c = k_h f B_m^{1\cdot 6} + k_e f^2 B_m^2 \qquad (26)$$

$$= \text{(hysteresis)} + \text{(eddy current loss)},$$

where k_h and k_e are constants.

PART II
PERMANENT MAGNETS

1. Permanently magnetized iron ring: remanence and coercive force.

If the magnetizing current in the winding of the closed iron ring of Fig. 86 is reduced from its maximum value to zero, the flux-density in the ring is reduced to the remanent flux-density, B_r (Fig. 93). The ring is then a "permanent" magnet, but the degree to which it will retain its remanent magnetism,

with use, will depend upon the magnitude of the *coercive force*, H_c.

Permanent magnets may be weakened by accidental application of a demagnetizing m.m.f. or mechanical shock (we assume that the temperature of the magnet is not raised above normal values), and the greater the coercive force, the less will be the effect of such disturbances, for the coercive force is a measure of the inter-atomic forces which oppose any change in the orientation of the atomic magnetic axes.

As the maximum m.m.f. gradient of the magnetizing coil (H_m, Fig. 93) is increased, a limit to the values of B_r and H_c is reached, so that no advantage is gained by extending H_m to very high values. In practice, the value of H_m necessary to produce maximum magnetization varies with the material to be magnetized, but a value of about 4×10^4 At/m (500 oersteds) is usually sufficient.

When figures for B_r and H_c are quoted for different permanent-magnet materials, the upper limiting values of these quantities are usually indicated. In this sense the *remanence* (B_r) and *coercivity* (H_c) are specific properties of a given material. The following table gives some representative values for the remanence and coercivity of various magnetic materials.

Material	B_r		H_c	
	Wb/m²	gauss	At/m	oersteds
Very soft pure iron	1·04	10,400	35	0·44
High-carbon steel (1·2 % carbon, quenched)	0·80	8,000	$4{\cdot}6 \times 10^3$	58
6 % Tungsten steel	1·10	11,000	$5{\cdot}6 \times 10^3$	70
35 % Cobalt steel	0·95	9,500	2×10^4	250
25 % Nickel, 13 % Aluminium	0·70	7,000	$3{\cdot}2 \times 10^4$	400
20 % Nickel, 10 % Aluminium, 10 % Cobalt	0·75	7,500	$4{\cdot}8 \times 10^4$	600

The product B_rH_c is often taken as the criterion of quality for a permanent-magnet material. This is roughly consistent with the condition for greatest economy discussed in Section 3, below.

2. The magnetized ring with an air-gap: demagnetization curves.

In the majority of practical applications of permanent magnets, their function is to maintain a magnetic field in an *air-gap*, and to deduce the approximate performance in such a case we may apply the circuital law.

Suppose that a uniform gap of length g (Fig. 112) is cut in a ring which gave, when complete, the hysteresis loop of Fig. 93. As a rough approximation, we may neglect leakage, so that we assume that the flux over any section A_i of the iron is

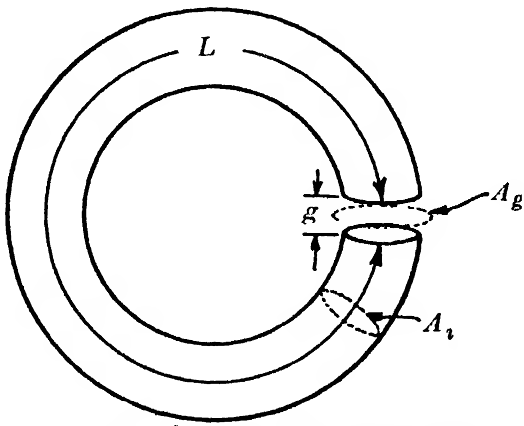


Fig. 112 Permanent magnet with air gap

constant, and that the same flux is uniformly distributed over an effective area A_g in the gap.

Since there is now no magnetizing coil on the ring, the circuital law gives

$$\oint H\,dl = H'L + H_g g = 0,$$

where H' = the m.m.f. gradient in the iron, and

$H_g = B_g/\mu_0$ = the m m.f. gradient of the field in the gap.

Hence $$H' = -\frac{B_g g}{\mu_0 L}. \qquad (27)$$

In order to interpret this result, let us analyse the conditions in a *complete* ring when the point P on the hysteresis loop

(Fig. 113) has been reached. The flux-density B has an m.m.f. gradient

$$H_t = \frac{B}{\mu_0},$$

in the same direction as the field B, but since the magnetizing coil sets up an m.m.f. gradient of magnitude $-H$ *opposing* B, it follows that the *actual* m.m.f. gradient provided by the oriented atomic magnets must be given by

$$H_i = H_t + H.$$

Thus when the magnetic history of the iron is such that conditions are given by the point P, it follows that, if the

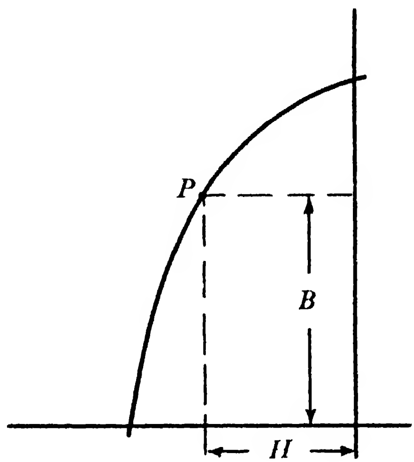


Fig. 113. Demagnetization curve

demagnetizing force $-H$ is removed and B maintained at the same value by introducing an air-gap, then the orientation of the atomic magnets is such as to provide an m.m.f. gradient, over the length L of the iron,

$$H_i = \frac{B}{\mu_0} + H,$$

where B and H are the numerical values of the co-ordinates of the point P in Fig. 113.

Since the length of the iron is L, the total m.m.f. provided by it is

$$m_i = H_i L = \frac{BL}{\mu_0} + HL.$$

The first term, BL/μ_0, is the m.m.f. used up in the length L of the iron, so that a surplus m.m.f., given by

$$m_s = HL,$$

exists which may be used up in the *air-gap*.

If the gap is of length g, and the mean flux-density in it is B_g, the m.m.f. required for the gap is

$$m_g = \frac{B_g}{\mu_0} g.$$

Equating this to the available surplus m.m.f., m_s, gives

$$H = \frac{B_g g}{\mu_0 L},$$

the arithmetic value of the H co-ordinate of the point P, and is identical with 4(27), which was obtained by using the circuital law.

The value of H given by 4(27) represents the m.m.f. gradient in the iron, corresponding to a flux-density *in the iron* of B, and B and H are connected by the hysteresis loop of a ring specimen. This is perfectly consistent with our definition of *H in iron* given in Section 10 of this chapter.

Now since H' as deduced above [4(27)] is *negative*, and B is *positive*, it follows that the performance of a permanent magnet can be obtained from a knowledge of that portion of the hysteresis loop in the $(B+,\ H-)$ quadrant, a portion which is known as the *demagnetization curve*. Typical demagnetization curves for various permanent-magnet materials are shown in Fig. 114. In order to apply this theory, we must be sure that the iron has been originally magnetized to a degree sufficient to leave the maximum remanence, B_r, in a closed ring.

Example. A fully magnetized ring of nickel-aluminium magnet steel (Curve C, Fig. 114) has a mean length of 30·5 cm. It is of circular section, of radius 1 cm., and has a uniform air-gap of length 0·5 cm. Making the usual correction for fringing at the gap, but neglecting leakage, find the mean flux-density in the gap.

Here $L = 0{\cdot}3$ metre, $g = 5 \times 10^{-3}$ metre,

The cross-section of the iron = 3·14 sq. cm.,

The effective area of the gap $= \pi(1{\cdot}25)^2 = 4{\cdot}91$ sq. cm.,

so that
$$B_g = B\frac{3{\cdot}14}{4{\cdot}91} = 0{\cdot}64B,$$
where B is the density in the iron.

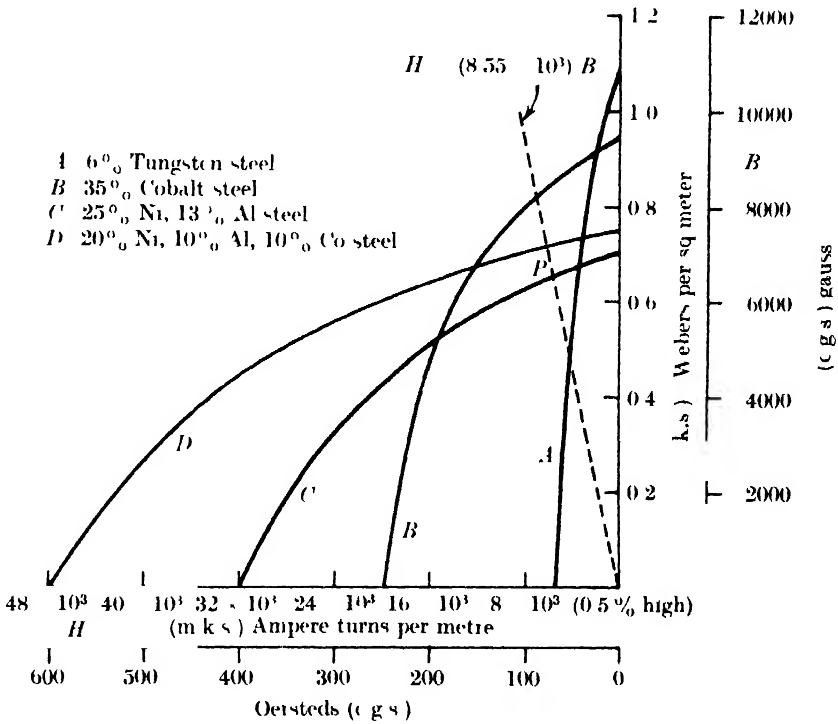


Fig. 114. Demagnetization curves

From 4(27):
$$-H = \frac{B_g g}{\mu_0 L} = \frac{0{\cdot}64B \times 5 \times 10^{-3}}{4\pi \times 10^{-7} \times 3 \times 10^{-1}}$$
$$= (8{\cdot}55 \times 10^3)\,B.$$

On Fig. 114 draw the straight line $-H = (8{\cdot}55 \times 10^3)\,B$. This cuts the Ni-Al curve (C) at the point P, where $B = 0{\cdot}65$ Wb/m².

Hence
$$B_g = 0{\cdot}64B = 0{\cdot}416\ \text{Wb/m}^2$$
$$= 4160\ \text{gauss}.$$

3. The condition for minimum volume of magnet material to set up a given flux-density in a given air-gap.

Let the required density in the gap be B_g, the effective area of the gap A_g, and its length g.

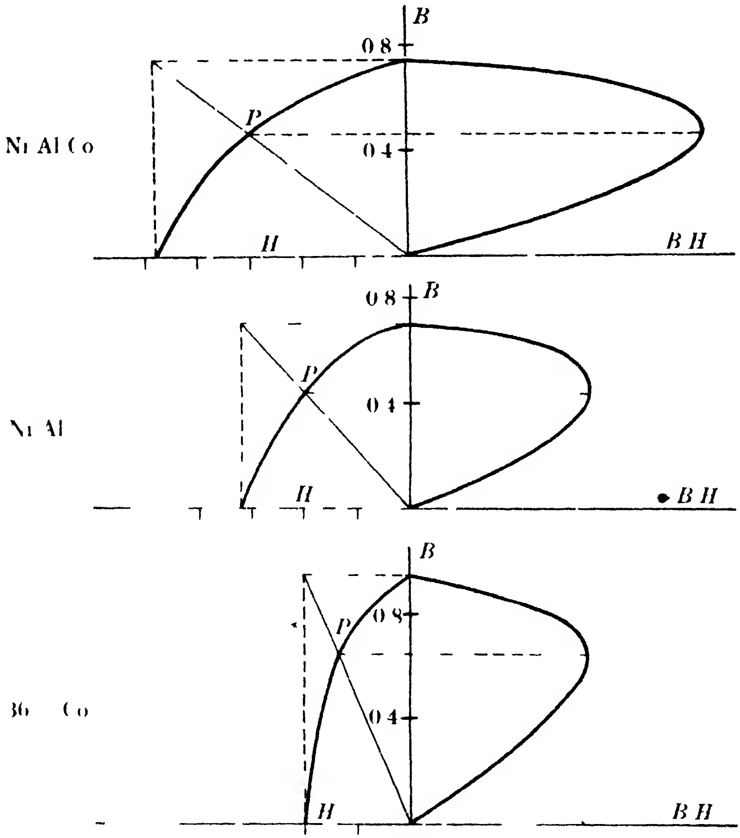


Fig. 115. Evershed's criterion

Then the energy stored in the gap is

$$W_g = \tfrac{1}{2} B_g H_g (A_g g)$$

but $$H_g = \frac{B_g}{\mu_0} = -H\frac{L}{g} \quad \text{from } 4(27),$$

and, neglecting leakage,

$$B_g A_g = BA,$$

where B is the density in the iron, A the cross-section of the iron, and $-H$ corresponds to B on the demagnetization curve.

Hence

$$W_g = \tfrac{1}{2}(-H)B(LA),$$

but LA is the volume of the permanent magnet, so that, if this is to be a minimum for a given gap-energy, the product $(-H)B$ must be a maximum. This relation is often called "Evershed's Criterion"* and has been verified experimentally.†

Values of the product $(-H)B$ are plotted, for three of the curves of Fig. 114, in Fig. 115. The values are plotted horizontally against ordinates of B, and the maximum values are marked by means of horizontal dotted lines.

Example of the design of a permanent magnet. A nickel-aluminium permanent-magnet steel (Curve C, Fig. 114) is to be used in a moving-coil instrument of the usual type with a cylindrical core between the pole-pieces. The core is 2·5 cm. in diameter and 3 cm. in axial length, and the angle of embrace of the pole-pieces is 120°. Each air-gap is 2 mm. long. The mean flux-density in the gaps is to be 0·2 w p.s.m. (2000 gauss). The total leakage may be represented by 50 % of the useful flux, and the m.m.f. required for the soft-iron yoke, pole-pieces and core may be taken as 10 % of that needed by the air-gaps. Neglect the effect of fringing at the gaps, and find the dimensions of the most economical permanent magnet (see Fig. 116).

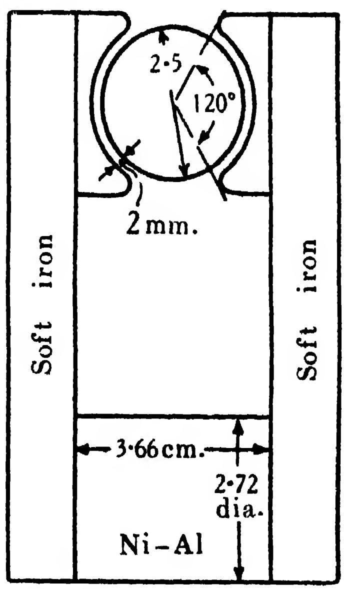


Fig. 116. Permanent magnet for moving-coil instrument

Solution. The mean area of the gap is

$$A_g = \frac{2\pi}{3} \times 1\cdot35 \times 3 = 8\cdot5 \,\text{sq. cm.} = 8\cdot5 \times 10^{-4} \,\text{sq. metre.}$$

The useful flux $= 8\cdot5 \times 10^{-4} \times 0\cdot2 = 1\cdot7 \times 10^{-4}$ weber.

The leakage flux $= 50\,\%$ of the useful flux $= 0\cdot85 \times 10^{-4}$ weber.

The total flux in the permanent magnet

$$= 2\cdot55 \times 10^{-4} \text{ weber} = \phi.$$

For minimum volume of magnet steel (Fig. 115, Ni-Al):

$$B = 0\cdot44 \text{ w.p.s.m.},$$

$$-H = 1\cdot92 \times 10^4 \quad \text{(see Fig. 114)},$$

hence the most economical area of the magnet is

$$A = \frac{\phi}{B} = \frac{2\cdot55 \times 10^{-4}}{0\cdot44} = 5\cdot8 \times 10^{-4} \text{ sq. metre}$$

$$= 5\cdot8 \text{ sq. cm.}$$

The m.m.f. required for the air-gaps is

$$m_g = \frac{B_g\, 2g}{\mu_0} = \frac{0\cdot2 \times 4 \times 10^{-3}}{4\pi \times 10^{-7}} = 636$$

Adding 10 % for the soft-iron parts, the total external m.m.f. required is

$$1\cdot1 \times 636 = 700 = (-H)\,L,$$

so that $\quad L = \dfrac{700}{1\cdot92 \times 10^4} = 3\cdot65 \times 10^{-2}$ metre $= 3\cdot65$ cm.

The magnet can be arranged as in Fig. 116, the permanent magnet consisting of a cylinder of radius 1·36 cm. and length 3·65 cm.

4. The representation of demagnetization curves by a rectangular hyperbola.

If the curve fits a rectangular hyperbola passing through the points $(B_r, 0)$ $(0, -H_c)$ its equation is (Fig. 117)

$$B = \frac{a(H + H_c)}{b + H + H_c}, \tag{28}$$

and its asymptotes are the lines

$$B = a,$$

$$H = -(H_c + b).$$

Putting $(b+H_c) = c$ the equation becomes

$$B = a\left(1 - \frac{b}{c+H}\right). \tag{29}$$

The remanence B_r is obtained by putting $H=0$. That is,

$$B_r = a\left(\frac{c-b}{c}\right) = \frac{a}{c}H_c. \tag{30}$$

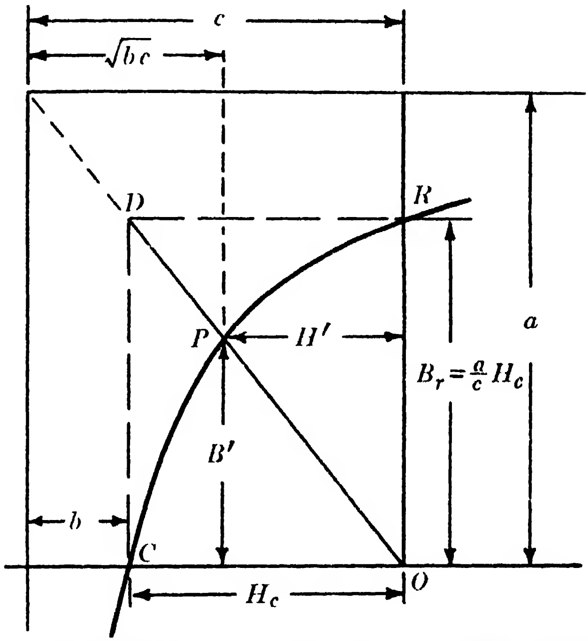


Fig. 117. Demagnetization curve represented by rectangular hyperbola

To find the constants a, b and c in 4(29), shift the origin to the point $C(0, -H_c)$, so that the abscissae become $H' = H_c + H$, where H represents the abscissae of the demagnetization curve, and has negative values. Plotting $(H_c+H)/B$ upon a base of (H_c+H) enables us to find a and b, and hence c (Fig. 118).

This is done, for the three curves B, C, and D of Fig. 114, in Fig. 118a, and it is seen that the correspondence is good. The equations for these demagnetization curves, in m.k.s. units, become

Curve B 35 % Cobalt steel

$$B = 1{\cdot}2\left[1 - \frac{6\times 10^3}{(2{\cdot}6\times 10^4)+H}\right]$$

Curve C. 25 % Nickel, 13 % Aluminium } steel

$$B = 1\cdot16\left[1 - \frac{2\cdot08 \times 10^4}{(5\cdot27 \times 10^4) + H}\right].$$

Curve D. 20 % Nickel, 10 % Aluminium, 10 % Cobalt } steel

$$B = 1\cdot16\left[1 - \frac{2\cdot67 \times 10^4}{(7\cdot47 \times 10^4) + H}\right].$$

To find the point where the product HB is a maximum. We have

$$BH = a\left(H - \frac{bH}{c+H}\right),$$

so

$$\frac{d(BH)}{dH} = \frac{a[H^2 + 2cH + c(c-b)]}{(c+H)^2};$$

this $= 0$ when $H = -c \pm \sqrt{bc}$.

If B is to be positive we must take the $+$ sign. That is, the point P (Fig. 117) of the demagnetization curve, at which the product BH is a maximum, has the co-ordinates

$$H' = -c + \sqrt{bc}$$

and

$$B' = a\left(1 - \sqrt{\frac{b}{c}}\right) \quad \text{(from 4(29))},$$

so that

$$\frac{B'}{-H'} = \frac{a}{c} = \frac{B_r}{H_c}. \tag{31}$$

Thus to find the point P we merely need to complete the rectangle O–R–D–C (Fig. 117), and the diagonal O–D cuts the demagnetization curve in the required point.

This construction is shown in Fig. 115 for each of the curves, and the points P so found are seen to agree very well with the values of B for which the product BH is a maximum. It will be noticed, however, that the graph of BH has a very blunt maximum, so that slight deviations from the true value of "the most economical B" will not result in any appreciable decrease in economy. Consequently, even though a particular

demagnetization curve may not fit equation 4(29), the above construction for obtaining the optimum values of B and H may still be relied upon.

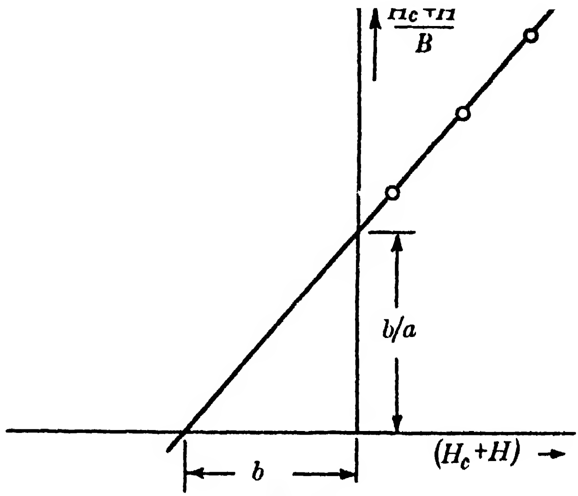


Fig. 118

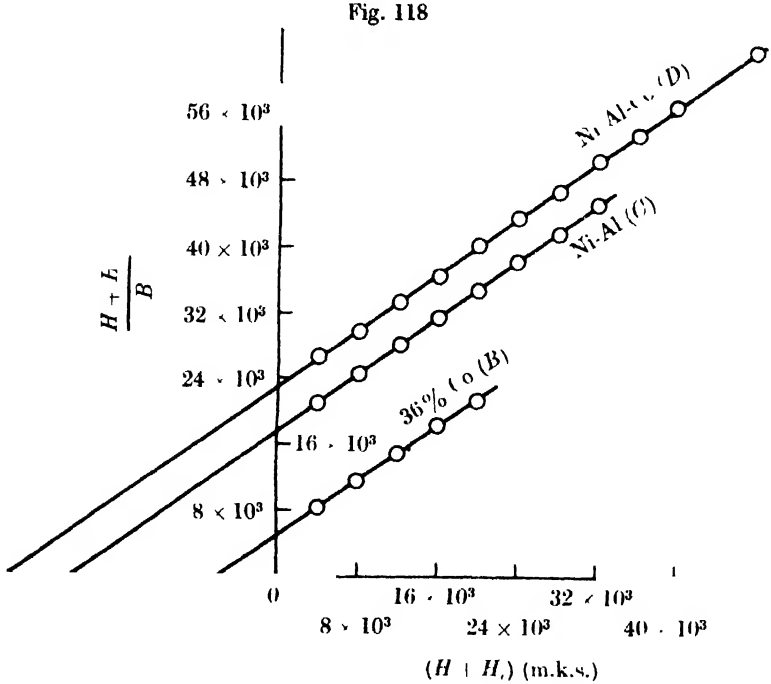


Fig. 118*a*

EXAMPLES, CHAPTER IV

1. A ring specimen of iron contains an air-gap 0·3 mm. in length: the mean diameter of the ring is 15 cm. How many ampere-turns must be put on the ring in order to produce an induction of 11,000 c.g.s. units in the iron?

The magnetizing current is then broken: what will be the value of the residual induction?

The relevant portion of the hysteresis loop is given by

H (At/m)	544	240	120	0	-40	-80	-120	-144
B (Wb/m^2)	1·1	1·05	1·0	0·88	0·8	0·67	0·42	0

(Neglect fringing and leakage.)

Ans. 517 ampere-turns, 0·26 Wb/m^2.

2. An iron ring, of uniform section, has an air-gap of length g, and the mean length of the iron is L. The mean flux-density in the gap is equal to kB, where B is the density in the iron, and k a constant less than unity.

If the B-H curve is given by Fröhlich's equation, 4(9), show that, neglecting leakage, the flux-density in the iron due to a magnetizing m.m.f., m, is given by the smaller root of the equation

$$B^2 - \left\{\frac{(m+bL)\mu_0}{kg} + a\right\} B + \frac{am\mu_0}{kg} = 0.$$

If the ring is of cast steel of circular section of 1 cm. radius, with a mean length $L = 50$ cm., and a gap $g = 2$ mm., find the flux-density in the gap when a current of 2 amperes flows in an exciting winding of 500 turns. Take $a = 1{\cdot}76$, $b \doteq 602$ for cast steel.

Ans. 0·523 Wb/m^2 (5230 gauss).

3. A ring is made up of stampings of the dimensions shown in Fig. 119 and is wound with 100 turns of wire. Assuming that the lines of magnetic flux follow paths similar to the dotted line, in which the curved portion at each corner is a circle struck from the point A as centre, find the distribution of flux and the average flux per sq. cm. across the section BB when the current in the wire is 2 amperes. The following are the magnetic data for the iron:

H (At/m)	320	400	560	800	1200	1600
B (Wb/m^2)	0·68	0·8	1·0	1·2	1·34	1·4

Explain why the lines of flux cannot in fact follow exactly the dotted path, and show by a sketch the way in which they deviate from it.

Ans. The flux-density varies from 1·22 Wb/m^2 at the inner edge to 0·9 Wb/m^2 at the outer edge, with a mean value of 1·05 Wb/m^2.

4. A horse-shoe magnet is formed from a bar of wrought iron 2 ft. long with a section of $1\frac{1}{2}$ sq. in. and a magnetizing coil of 1500 turns is

wound over the core. If the magnet is required to lift a load of 200 lb., find the minimum exciting current required, assuming that the two air-gaps of contact with the load are 1 mm. long and each has an area of $1\frac{1}{2}$ sq. in. Take the (relative) permeability of the wrought iron as 700 and the reluctance of the magnetic circuit through the load as 50 % of the reluctance of the horse-shoe (excluding the air-gap).

(London, External B.Sc., 1935.)

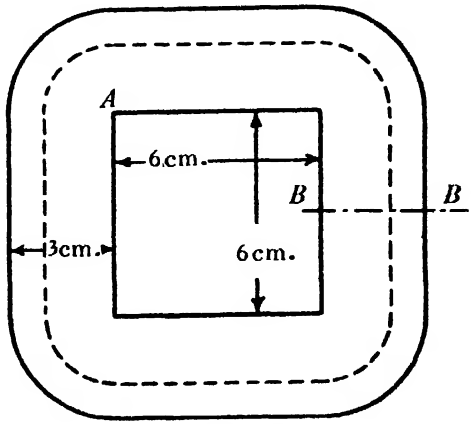


Fig. 119

Solution.

The force of attraction per gap = 100 lb. $= \frac{100}{0\cdot2247} = 445$ newtons.

The length of iron: $L = 2$ ft. $= \frac{2}{3\cdot281} = 0\cdot61$ metre.

Length of each gap: $g = 10^{-3}$ metre.

Area of iron and gaps: $A = 1\frac{1}{2}$ sq. in. $= 9\cdot68 \times 10^{-4}$ sq. metre.

Thus the force per gap, in newtons per sq. metre,

$$= \frac{445 \times 10^4}{9\cdot68} = 4\cdot6 \times 10^5 = \frac{B^2}{2\mu}, \text{ [equation 4(22)]}$$

whence $\qquad B^2 = 1\cdot55, \quad$ and $\quad B = 1\cdot075\,\text{Wb/m}^2$.

(Neglecting leakage, this is the density in both iron and gaps.)

H, in the magnet,

$$= \frac{B}{\mu\mu_0} = \frac{1\cdot075 \times 10^7}{4\pi \times 700} = 1220.$$

Thus the ampere-turns for the magnet-iron $= 1220 \times 0\cdot61 = 746$

The ampere-turns for the gaps $= \frac{2Bg}{\mu_0} = 1713$

The ampere-turns for the load = 50 % that for the iron $= 373$

Total: 2832

The exciting current $= \frac{2832}{1500} = 1\cdot89$ amperes.

5. Calculate the hysteresis loss per hour in a sample of iron weighing 10 lb. for which the hysteresis loop has an area equivalent to 1200 ergs per cubic cm. (120 joules per cubic metre), when subjected to alternating magnetization of frequency 50 cycles per second. Take the density of the iron as 7·5. (London, External B.Sc., 1934.)

Ans. 3·64 watt-hours or $1{\cdot}31 \times 10^4$ joules.

6. The materials and leading dimensions of the magnetic circuit of a certain eight-pole generator are as follows:

Portion	Material	Area	Length
Yoke	Cast steel	290 sq. cm.	70 cm.
Pole cores	Mild steel	540 sq. cm.	32 cm.
Pole shoe	Laminated silicon steel	25·4 cm. wide 30·5 cm. axial length	
Air-gap	—	—	0·43 cm. (uniform)
Teeth	Silicon steel	2·5 cm. wide at top 30·5 cm. axial length	4 cm. deep
Iron behind teeth	Silicon steel	420 sq. cm.	30 cm.

There are seven teeth in the polar arc; the slot width is 1·2 cm Relevant magnetic details of the materials are as follows (m.k.s.):

Cast steel	$B = 1{\cdot}6$ $H = 3200$	$B = 1{\cdot}5$ $H = 1600$	$B = 1{\cdot}3$ $H = 800$
Mild steel	$B = 1{\cdot}6$ $H = 4800$	$B = 1{\cdot}5$ $H = 3200$	$B = 1{\cdot}4$ $H = 2400$
Silicon steel	$B = 1{\cdot}4$ $H = 800$	$B = 1{\cdot}2$ $H = 480$	$B = 0{\cdot}8$ $H = 240$

Estimate the number of shunt ampere-turns per pair of poles to produce a flux across the air gap of $6{\cdot}5 \times 10^{-2}$ webers. Explain clearly your method of allowing for fringing, etc.

(Oxford, 1933, amended to m.k.s.)

Ans. Approximately 9000 ampere-turns if leakage coefficient is 1·2.

7. The magnetization curve of a certain iron specimen is as follows:

Ampere-turns per in.	0	10	20	30	40	50	60
Kilo-lines per sq. in.	0	61	84	94	98·7	101	103

A core of this material has the form shown in the sketch (Fig. 120). The centre leg has an area of 3 sq. in., the areas of the outer legs and horizontal portions being 2 sq. in. Calculate the ampere-turns required on the centre leg if the flux-density in the air-gaps is to be 70 kilo-lines per sq. in. Neglect leakage but allow for fringing. *Ans.* 1990.

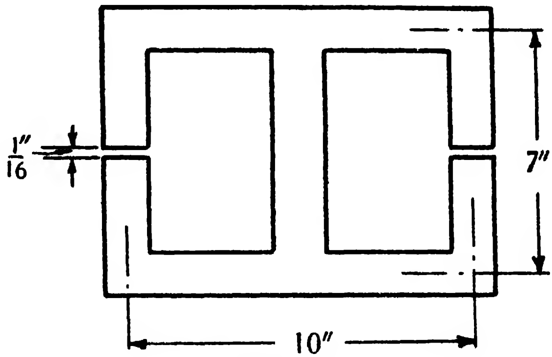


Fig. 120

8. The demagnetization curve of a permanent-magnet steel is given by equation 4(29), and a magnet is to maintain a useful external field in which the energy stored is W. If the flux in the magnet is equal to k_L times the useful flux, show that the minimum volume of magnet material required is

$$\frac{2Wk_L}{ca\left(1-\sqrt{\frac{b}{c}}\right)^2},$$

where a, b and c are the constants in equation 4(29).

9. A cylindrical permanent bar-magnet has a length L and a radius R. Assuming that all the flux passes through the plane ends of the magnet, and in the surrounding air has the same configuration as that issuing from the ends of an air-cored solenoid of the same dimensions, show that the "operating point" on the demagnetization curve is given roughly by

$$\frac{B}{-H}=\frac{L\mu_0}{8\pi R}, \quad \text{provided } L \gg R.$$

Note. From Ex. 11, Chapter III, the m.m.f. required to maintain the field, external to the coil, between the ends of a long solenoid is approximately equal to $IN\frac{R}{L}$, and the flux maintained by this m.m.f. is approximately 50 % of that through the central section of the solenoid. Hence show that the reluctance of the air-path between the ends of the bar-magnet is roughly equal to $\frac{8}{\mu_0 R}$.

10. Describe a method of finding a B-H cyclic curve for a sample of sheet steel.

The table below gives the loss (W) in a specimen of sheet steel for various values of the maximum flux-density (B) and the frequency (n):

B	n	W
5000	20	6·2
	40	15·6
	60	28·0
9000	20	16·8
	40	43·4
	60	80·0

(B is in gauss). Show that these figures indicate that the hysteresis loss in the specimen is proportional to nB^x and deduce the value of x.

(Cambridge, A, 1931.)

Ans. 1·63.

11. Redesign the magnet system for the moving-coil instrument described in the example in Section 3, Part II, of this chapter (p. 243) if the permanent magnet is to be

(a) Tungsten steel.
(b) 35 % cobalt steel.
(c) Al-Ni-Co steel.

(See Fig. 114 for demagnetization curves.)

12. A ring, made of iron whose magnetic properties are given below, has a mean diameter of 15 cm. and a cross-section of 8 sq. cm. A slit 2 mm. wide is made in the ring at one point. Calculate the ampere-turns required on the ring to give a pull of 320 newtons tending to close the slit.

B (Wb/m²)	0·8	1·0	1·2
H (At/m)	250	360	530

Ans. 1760.

13. An iron ring of sectional area 2 sq. in. and axial length 20 in. is made of material whose magnetization curve is as follows:

Ampere-turns per in.	6	10	20	30
Kilo-lines per sq. in.	41	62	84	94

The ring has two windings, one of 20 turns and one of 100 turns. Assuming that the coefficient of coupling is unity, calculate the coefficient of mutual inductance, in milli-henries, when the total flux in the ring is (a) 188 kilo-lines, and (b) 124 kilo-lines.

Ans. 6·3 and 12·4 milli-henries.

14. Show that the power loss due to eddy currents in an armature of given dimensions is proportional to the square of the product of the maximum induction in the armature stampings, the number of cycles per second, and the thickness of the stampings.

The iron loss in the armature of a direct-current generator for a given excitation is found to be 1000 watts at 800 r.p.m. and 300 watts at 300 r.p.m. What is the eddy current loss at the higher speed? Estimate the total iron loss at 750 r.p.m., the excitation of the machine having

been so changed that the e.m.f. is the same as it was at 800 r.p.m. with the original excitation. (Cambridge, A, 1932.)

Ans. 320 watts, 1030 watts.

15. The flux in a closed iron ring is given by

$$\phi = \frac{am}{b+m},$$

where m is the m.m.f. of the magnetizing winding.

A direct-current m.m.f., m_0, is applied and causes a flux Φ_d, and an alternating m.m.f. is then superposed upon m_0 such that the flux in the ring now has a sinusoidal alternating component given by $\Phi \sin \omega t$. If hysteresis and eddy current loss are negligible, and if the direct-current component of m.m.f. is still m_0, show that the superposition of the alternating flux causes the constant component of flux to decrease to the value

$$d = a - \sqrt{(a-\Phi_d)^2 + \Phi^2}, \quad \text{provided that } \Phi < d.$$

Hence show that the direct-current m.m.f. necessary to cause a constant component of flux d, when a sinusoidal flux component $\Phi \sin \omega t$ is present, is

$$m_0 = b\left\{\frac{a}{\sqrt{(a-d)^2 - \Phi^2}} - 1\right\}.$$

Further, show that the root mean square value of the alternating m.m.f. is

$$m_a = (b+m_0)\left\{\sqrt{1 + \frac{\Phi^2(b+m_0)^2}{a^2b^2}} - 1\right\}^{\frac{1}{2}},$$

and that the harmonic analysis (Fourier series) for m_a is

$$m_a = \frac{abp}{(a-d)}[B_1 \sin \omega t - B_2 \cos 2\omega t - B_3 \sin 3\omega t + B_4 \cos 4\omega t + B_5 \sin 5\omega t - \text{etc.}],$$

where

$$p = \frac{\Phi}{(a-d)},$$

$$B_1 = \frac{2}{p^2}\left[\frac{1}{\sqrt{1-p^2}} - 1\right],$$

$$B_2 = \frac{2^2}{p^3}\left[\frac{1-\frac{p^2}{2}}{\sqrt{1-p^2}} - 1\right],$$

$$B_3 = \frac{2}{p}B_2 - B_1,$$

$$B_4 = \frac{2}{p}B_3 - B_2,$$

....

and, in general,

$$B_n = \frac{2}{p}B_{n-1} - B_{n-2}.$$

Again provided that $\Phi < d$. How may these results be applied to the case of anode-bend rectification by a triode valve?

(*Note.* This example is above the usual standard of undergraduate work.)

CHAPTER V

ELECTRO-MAGNETIC WAVES. THE VECTOR POTENTIAL OF THE ELECTRIC CURRENT AND ITS USES

PART I

ELECTRO-MAGNETIC WAVES, USING THE MAGNETIC FIELD CONCEPT

1. The problem.

When a steady current flows in a wire, the attendant magnetic field may be calculated upon the basis of equation 3(4) alone

$$B_0 = \mu_0 \frac{I\,\delta l}{4\pi r^2} \sin\alpha, \tag{3(4)}$$

but when the current changes an electric field is induced in the surrounding space which, changing, is attended by a magnetic field which may be considered as that of the *displacement* currents, while 3(4) gives the field due to the *conduction* current in the wire

The induced electric field will then be related to the total magnetic field by

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d\phi}{dt}, \tag{3·26}$$

and the magnetic field obeys the relation

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 \frac{d\psi}{dt}, \tag{3(25)}$$

so long as the path integrated around does not link with the conduction current. A simultaneous solution of 3(25) and 3(26) will give us the conditions at a point where the only magnetic field present is that due to displacement currents, a condition which is true at great distances from a high-frequency current circuit. The fields are then said to form an electro-magnetic wave.

2. Three-co-ordinate equations for the magnetic field of a displacement current (i.e. of a changing electric field).

Let E and B (due to E) at any point P in a homogeneous medium of dielectric constant K and relative permeability μ each be resolved into three components parallel to the axes of x, y and z (Fig. 121). Apply equation 3(25) to a small rectangular

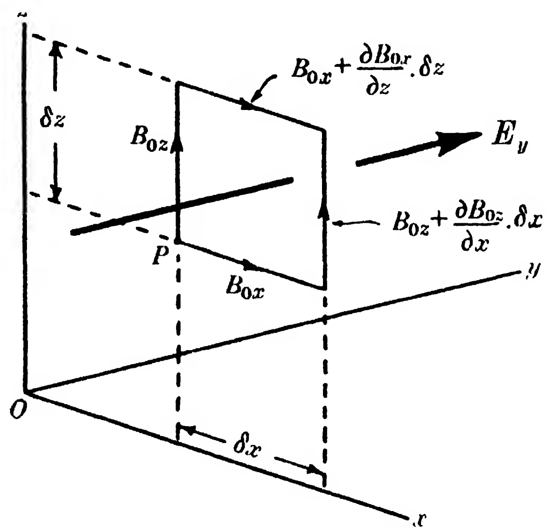


Fig. 121

circuit of sides δx and δz, in the z-x plane. Then E_y is the component of E normal to this circuit, and

$D_y = K\epsilon_0 E_y =$ the y-component of the displacement density,

$\psi_y = D_y\, \delta x\, \delta z =$ the displacement through the rectangle

$= K\epsilon_0 E_y\, \delta x\, \delta z,$

so that 3(25) becomes

$$B_{0z}\,\delta z + \left(B_{0x} + \frac{\partial B_{0x}}{\partial z}\,\delta z\right)\delta x - \left(B_{0z} + \frac{\partial B_{0z}}{\partial x}\,\delta x\right)\delta z - B_{0x}\,\delta x$$
$$= \mu_0 K\epsilon_0 \frac{\partial E_y}{\partial t}\,\delta x\,\delta z,$$

or
$$\frac{\partial B_{0x}}{\partial z} - \frac{\partial B_{0z}}{\partial x} = K\mu_0\epsilon_0 \frac{\partial E_y}{\partial t} = \frac{K}{c^2}\frac{\partial E_y}{\partial t}.$$

The flux-density $B = \mu B_0$, so that

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \frac{\mu K}{c^2}\frac{\partial E_y}{\partial t}, \tag{1}$$

and by a similar treatment on elementary rectangles in the x-y and y-z planes:

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{\mu K}{c^2}\frac{\partial E_z}{\partial t}. \tag{2}$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\mu K}{c^2}\frac{\partial E_x}{\partial t}. \tag{3}$$

3. Three-co-ordinate equations for the electric field of a changing magnetic field.

Now apply equation 3(26) in a similar way to a rectangle of sides δx, δz in the z-x plane (Fig. 122). The flux through the rectangle is

$$\phi_y = B_y \delta x \, \delta z,$$

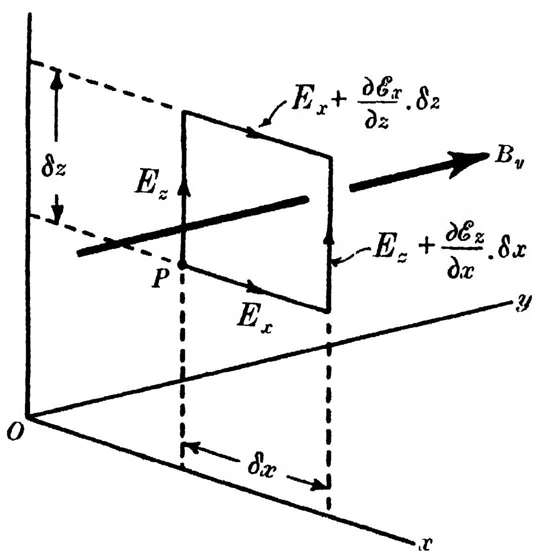


Fig. 122

so that

$$E_z \delta z + \left(E_x + \frac{\partial E_x}{\partial z}\delta z\right)\delta x - \left(E_z + \frac{\partial E_z}{\partial x}\delta x\right)\delta z - E_x \delta x = \frac{\partial B_y}{\partial t}\delta x \, \delta z.$$

or

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t}, \tag{4}$$

and two similar equations:

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t}, \tag{5}$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t}. \tag{6}$$

4. The wave-equation for the simplest case.

Suppose that the field distribution is two-dimensional. That is, that there is no variation in the fields in the direction of the y-axis, and that there is an infinite and perfectly conducting plane surface in the x-y plane. This means that the electric field cannot have components in the x or y directions, so that

$$\frac{\partial}{\partial y}=0$$

and

$$E_x=E_y=0.$$

Further, assume that B is due entirely to displacement currents. Then of the above six equations, all but (2) and (4) disappear. Equation 5(2) becomes

$$\frac{\partial B_y}{\partial x}=\frac{\mu K}{c^2}\frac{\partial E_z}{\partial t}, \tag{7}$$

and equation 5(4):

$$\frac{\partial E_z}{\partial x}=\frac{\partial B_y}{\partial t}; \tag{8}$$

from 5(7):

$$\frac{\partial^2 B_y}{\partial x\,\partial t}=\frac{\mu K}{c^2}\frac{\partial^2 E_z}{\partial t^2},$$

and from 5(8):

$$\frac{\partial^2 B_y}{\partial x\,\partial t}=\frac{\partial^2 E_z}{\partial x^2},$$

whence, eliminating B_y,

$$\frac{\partial^2 E_z}{\partial x^2}=\frac{\mu K}{c^2}\frac{\partial^2 E_z}{\partial t^2},$$

or

$$v^2\frac{\partial^2 E_z}{\partial x^2}=\frac{\partial^2 E_z}{\partial t^2},\quad \text{where } v^2=\frac{c^2}{\mu K}. \tag{9}$$

The solution of equation 5(9) (the "wave-equation") is of the form

$$E_z = f(x - vt) + f'(x + vt),^*$$

a particular case being

$$E_z = A \sin p(x - vt + \alpha), \tag{10}$$

where A, p and α are constants.

To interpret the physical meaning of 5(10) we notice that:

(*a*) At a fixed point, x = constant, the field varies sinusoidally with time.

(*b*) At any given instant, t = constant, the field varies sinusoidally with the distance x.

(*c*) If $x/t = v$, E_z is constant. That is, if we imagine ourselves to be moving along the axis of x with velocity v, we shall be situated in an unchanging electric field

We therefore say that an *electro-magnetic wave* travels along the axis of x with velocity

$$v = \frac{c}{\sqrt{\mu K}}.\dagger \tag{11}$$

From 5(7) and 5(10) we get, for B_y,

$$B_y = -\frac{\mu K v}{c^2} E_z$$

$$= -\frac{E_z}{v}, \tag{12}$$

and from 5(5) and 5(6)

$$B_x = B_z = 0.$$

Thus at any fixed point P (Fig. 123), in the path of the "wave", there exist alternating electric and magnetic fields,

* A general deduction from 5(9) is that, if E and B suffer a change due to a change in the parental currents, then this change does not take place simultaneously at all points in the field. If a point is distant x from the source of the disturbance, the change in the fields at the point takes place at an instant x/v seconds later than the corresponding change at the source.

† In experiments on the refraction of light in transparent insulators ($\mu = 1$), it is found that this equation does not always give results consistent with the measured value of K. This is because, owing to the inertia of electrons and dipole molecules, K is not independent of frequency.

E_z and B_y, mutally perpendicular to each other and to the direction of propagation.

Now the direction of the vectors in Fig. 123 is such that (E_z, B_y, v) form a right-handed set, a relation which is identical with that shown in Figs. 47 and 49 (Chapter III). Further, if we put $\alpha = \pi/2$ and $v = c$ in each of the equations 3(6) and 2(11a) (Chapter III, Section 2), these equations become, for free space,

$$E = Bc \quad \text{and} \quad B = \frac{E}{c},$$

which results are identical with 5(12).

It thus appears that the phenomena observed by a stationary

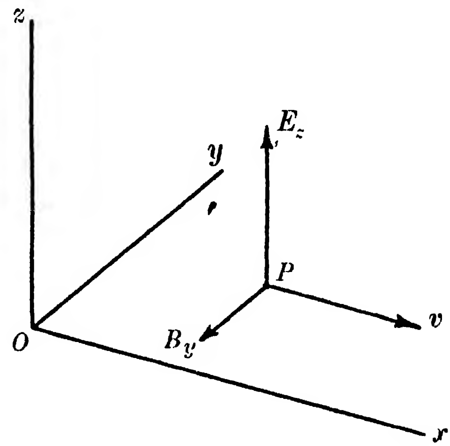


Fig. 123. Fields in electro-magnetic wave

observer at P (Fig. 123) *could* be explained by either of the two hypotheses:

Either (A) That a magnetic field B_y is moving in the direction of the wave with velocity c, and hence an electric field $\dot{E}_z$ exists at the point.

Or (B) That an electric field E_z is moving in the direction of the wave with velocity c, and hence a magnetic field B_y exists at the point.

The existence of such a reciprocal relation is, however, no proof that there is any motion of the fields, and if we are tempted to think that a field *is* moving we are immediately

faced with the problem of deciding *which* component, E or B, is moving. Upon the application of our definition of moving fields (Chapter III, Section 1) we see that there can be no motion of either field in a sense which has any physical meaning. At the point P, which is fixed relatively to the system of currents which causes the disturbance, we can say only that B and E are changing in a periodic manner, with a frequency

$$f = \frac{pv}{2\pi} \text{ cycles per sec.,} \tag{13}$$

which is dependent upon the frequency of the alternating currents to which the wave is due.

A useful analogy is that of the ripples caused on the surface of a pool when a pebble is dropped in. Watching the disturbance from the shore, we see a series of circular elevations and depressions grow out from the centre at a uniform rate, the distance between consecutive elevations being constant. It might appear to us at first sight that the water is actually moving, but then we notice that a small piece of wood, floating upon the surface, is not carried along with the disturbance but merely rises and falls, thus convincing us that the elevation of the surface varies periodically, and that the "motion" is a motion of the disturbance or "wave" and not of the water.

The case which we have considered is an ideally simple one, chosen because of the simplicity of the wave-equation, 5(9). The propagation of an electro-magnetic wave can never be so simple, but the above result may be accepted as showing the following general properties:

(*a*) The speed of propagation.

(*b*) The relative values of the electric and magnetic components.

(*c*) The relative directions of the vectors **E**, **B** and **v**.

The problem of relating the magnitude of the fields, at a point in a wave, to the system of changing currents to which it is due, is considered in an introductory manner in Part II or this chapter.

Let us take an instantaneous "snapshot" of a wave. To do this we put $t=$ constant in equation 5(10), and we get the result

$$E_z = A \sin p(x+\beta),$$

where β is a constant.

The field intensity thus varies with distance in a sinusoidal manner, as shown in Fig. 124. The length of a complete cycle is called the *wave-length*, λ, and is given by

$$p\lambda = 2\pi,$$

or $$\lambda = \frac{2\pi}{p}, \tag{14}$$

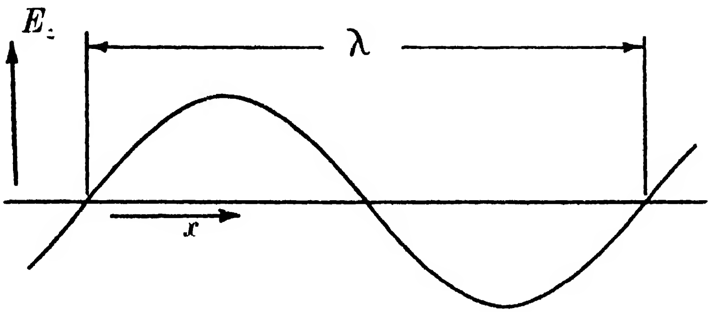


Fig. 124. Wave-length

but from 5(13) $$\frac{2\pi}{p} = \frac{v}{f},$$

so that $$\lambda f = v, \tag{15}$$

that is,

(Wave-length) × (frequency) = Velocity of propagation.

In free space $K = \mu = 1$, so that

$$v = c \doteqdot 3 \times 10^8 \text{ metres per sec.},$$

and equation 5(15) gives the relation between wave-length (in metres) and frequency (in cycles per second), so well known in radio communication.

5. The induction of an e.m.f. in a conductor by an electro-magnetic wave.

A method of thinking of the total effects of an alternating current in a circuit which may prove helpful is to consider

the magnetic field at any point near the circuit as being made up of two components:

(a) The field of the conduction currents.
(b) The field of the displacement currents.

If a second circuit is situated in this magnetic field, an e.m.f. will be induced which again we can regard as having two components:

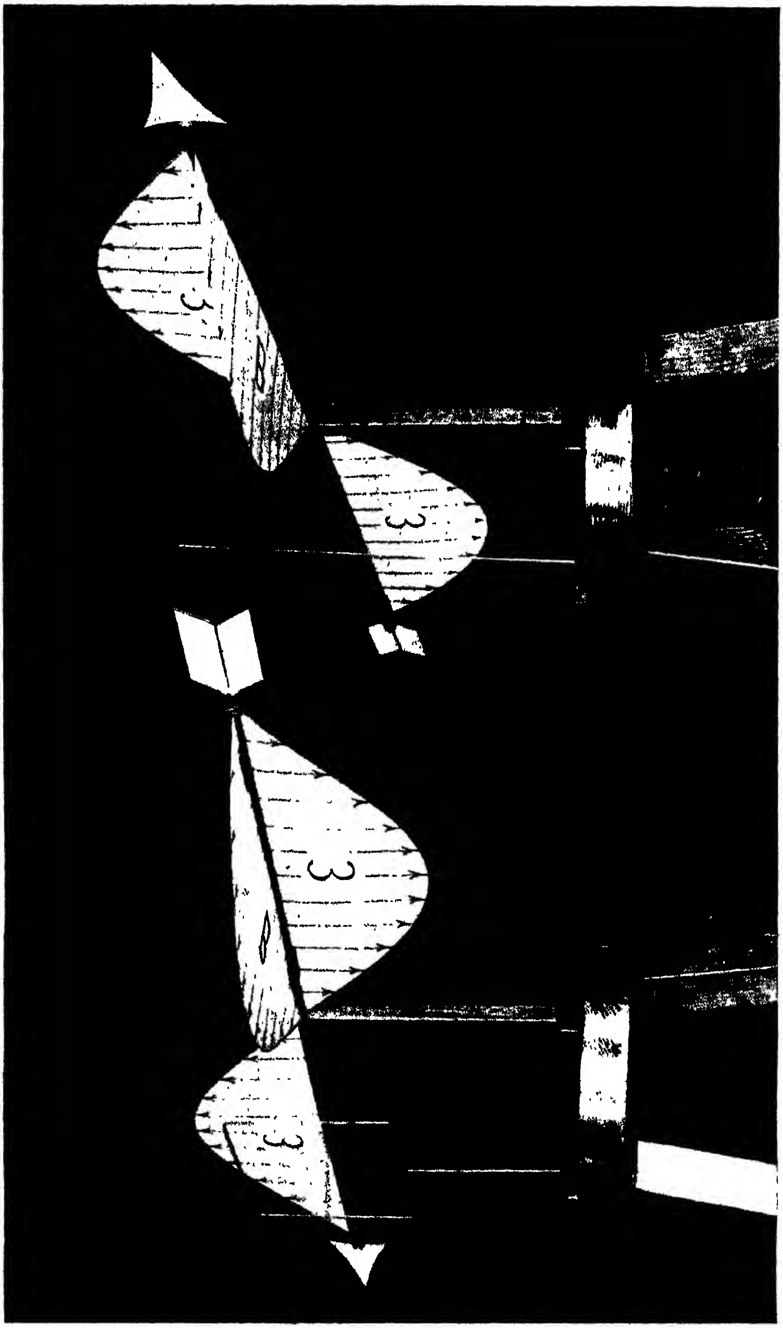
(a) That induced by the field of the conduction currents in the first circuit. This will be given by $e = -M\frac{di}{dt}$, where M is the coefficient of mutual inductance, as calculated for the conduction current i only.
(b) That induced by the field of the displacement currents; that is, by the electro-magnetic wave propagated from the first circuit.

The field of the conduction currents is independent of frequency, and rapidly decreases as we move away from the current circuit. The field of the wave, however, does not decrease so rapidly and at great distances from the circuit is the only field that is of practical importance. Further, its magnitude increases rapidly as the frequency of the parental current increases. In what follows we shall confine the discussion to this latter component of e.m.f., upon which radio communication depends.

In explaining the e.m.f. induced in a receiving antenna by a wave, there is perhaps a tendency to fall into the error of thinking that both E and B result in an induced voltage. We sometimes find it stated that the e.m.f. induced in a single wire is due to E, while that induced in a loop antenna is due to B. This of course is erroneous, since B is merely a different aspect of E, and the truth is that the e.m.f. in *any* type of antenna can be calculated by using *either E or B*, but *not* both.

Since an e.m.f. is the line-integral of an electric field intensity, it follows that the most *direct* method of calculation is that making use of the "electric vector", E, and that if we use the "magnetic vector", B, to calculate the e.m.f. we are really using the relation of equation 5(12) to give us E.

PLATE I



MODELS TO DEMONSTRATE ELECTRO-MAGNETIC WAVES

The author has found the models shown in Plate I, which are made of wood and cardboard, to be useful in explaining both the e.m.f.'s induced in receiving antennae and the elementary principles of directional transmitting aerials. The arrow shows the direction of propagation of the wave, and the sinusoidally shaped fins denote the direction of the field vectors E and B. To use the model, we imagine it travelling with the velocity of the wave in the direction of the arrow. Then the magnitudes of the fields at any fixed point in the path of the wave are given, relatively, by the lengths of the arrows on the fins which happen to be at the point at the instant considered. The arrows on the fins do not, of course, represent "lines of force" moving with velocity v.

With these models it is readily seen that a single straight wire must be arranged vertically for an e.m.f. $e = EL$ to be induced, the fact that e is also equal to BLv being merely a consequence of equation 5(12). A loop must be arranged with its plane parallel to the direction of propagation, and if it is rectangular, with sides vertical and horizontal, the e.m.f. will be given by $e = (E_1 - E_2)L$, where E_1 and E_2 are the values of the electric field intensity at the two vertical sides, and L is the length of these sides. The e.m.f. is also given by $(B_1 - B_2)Lv$ or by $-d\phi/dt$, where ϕ is the surface integral of B over the area of the loop.

6. The energy carried by an electro-magnetic wave.

In terms of E and B, the volume densities of energy in the field are

$$\frac{K\epsilon_0 E^2}{2} \text{ in the electric field} \qquad 1(21)$$

and

$$\frac{B^2}{2\mu\mu_0} \text{ in the magnetic field.} \qquad 3(49a)$$

Now

$$B^2 = \frac{\mu K}{c^2} E^2, \quad \text{from } 5(12)$$

and

$$\frac{1}{\epsilon_0 \mu_0} = c^2,$$

whence the energy of the magnetic field is seen to be equal to

$$\frac{\mu K E^2}{2c^2\mu\mu_0} = \frac{K\epsilon_0 E^2}{2} \text{ per unit volume,}$$

which is equal to the volume density of energy in the electric field.

Thus the total energy, per unit volume, in an electro-magnetic wave, is

$$K\epsilon_0 E^2 = B^2/\mu\mu_0 = ED = HB, \tag{16}$$

and is shared equally by the electric and magnetic components.

7. The rate of energy flow: Poynting's vector.

The energy per unit volume of an electro-magnetic wave is, by 5(16),

$$W = K\epsilon_0 E^2.$$

Now replace one of the factors E by

$$E = Bv = \mu\mu_0 H \frac{c}{\sqrt{\mu K}}$$

so that

$$W = \frac{\sqrt{\mu K}}{c} EH = \frac{EH}{v}. \tag{17}$$

Now if we assume that this energy travels with velocity v in the direction of propagation,* it follows that the rate of

* The energy $\frac{1}{2}LI^2$ in the magnetic field of a conduction current I returns to the conducting circuit when the current collapses, since an e.m.f. $-L\,di/dt$ is induced in the circuit which will continue to drive current through the circuit resistance until all this magnetic energy has been converted into heat (consider the theory of Section 21, Chapter III). The energy in an electro-magnetic wave, however, appears to travel away from the current-circuit, and most of it is lost in space. The "mechanism", in terms of electric and magnetic fields, may be roughly described as follows: when the magnetic field of the conduction current collapses, it induces e.m.f.'s around closed loops in space as well as in the current-circuit. As these induced electric fields grow, they take energy from the collapsing magnetic field. This energy is then supposed to be handed on to new magnetic fields, induced by these changing electric fields (displacement currents); the induction of new fields, and the handing-on of energy, spreads outwards from

energy flow through unit area, normal to the direction of the wave, is

$$p = EH. \tag{18}$$

This is a particular instance of *Poynting's Theorem*,* which states that the direction of energy flow in an electro-magnetic field is perpendicular both to E and to H, and that its rate is given by

$$p = EH \sin \alpha, \text{ or } \mathbf{p} = (\mathbf{E} \times \mathbf{H}) \text{ per unit area,} \tag{18a}$$

where α is the angle between E and H (or B).

Further, from Fig. 123 it is seen that if p is regarded as a vector quantity, giving both direction and magnitude of the power-density, then the vectors $(\mathbf{E}, \mathbf{H}, \mathbf{p})$ form a right-handed set.

Example. As a simple example, suppose a straight wire to be connected, normally, to two parallel infinite conducting planes (Fig. 125, a) and that it carries a steady current I amperes. If L is the length of the wire, the electric field intensity is everywhere parallel to the wire, and is given by

$$E = \frac{V}{L} \text{ volts per metre,}$$

where V is the p.d. in volts between the planes.

The m.m.f. gradient of the magnetic field of I, at any point distant r from the wire, is given by

$$2\pi r H = I \quad \text{or} \quad H = \frac{I}{2\pi r}.$$

the circuit as a disturbance, or wave, which travels at the speed $c = \dfrac{c}{\sqrt{\mu K}}$.

This picture of the "mechanism" of a wave is analogous to the case of waves in an elastic solid body, such as a steel bar. Unfortunately the whole process is merely hypothetical, and in the light of modern quantum theory we must admit that we are totally ignorant of the real "mechanism" by which energy is radiated in an electro-magnetic wave.

* J. H. Poynting, "On the Transfer of Energy in the Electro-magnetic Field", *Phil. Trans.* CLXXV (1884), p. 343.

Hence the "Poynting Vector" is

$$p = EH = \frac{VI}{2\pi rL} \text{ watts per sq. metre}$$

and is directed *normally towards the wire* (Fig. 125, *b*), and the total power flowing into the wire is that passing through the surface of a cylinder of radius r and length L. That is, the power necessary to maintain the current is equal to

$$P = p2\pi rL = VI \text{ watts.}$$

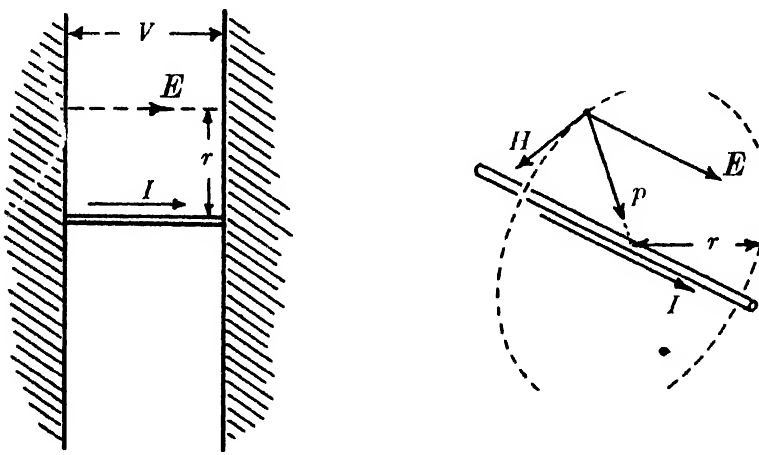


Fig. 125

Thus, according to Poynting's theory, the energy supplied to a conductor carrying current does not flow *through* the wire, but through the surrounding electro-magnetic field. Further, whatever energy enters a wire through its surface is lost as heat,* and the mechanical energy developed by an electric motor must flow from the generator through the *insulating medium* surrounding the connecting leads.†

When Poynting's theorem was first published, physicists regarded the aether as something having physical reality, in which energy could be stored in much the same way as in a

strained steel bar or a rotating flywheel. Poynting's work immensely strengthened this belief, as it gave an exact picture of the way in which electro-magnetic energy travelled from point to point in the strained medium. In the light of the modern theories of relativity and quantum mechanics, the belief in the physical existence of "magnetized" or "electrified" space cannot logically survive, at least in the old form, and the equations of energy storage and energy flow in the electro-magnetic field should be regarded merely as useful mathematical expressions by which we can perform certain calculations. Sir James Jeans, in *The Mysterious Universe*, says:

> We find that the attempt to parcel out energy amongst the different parts of space leads to an ambiguity which cannot be resolved. It seems natural to suppose that our attempt is a misguided one, and that the partition of energy through space is illusory.
>
> And again, the attempt to regard the flow of energy as a concrete stream always defeats itself. With a stream of water, we can say that a certain particle of water is now here, now there; with energy it is not so.* The concept of energy flowing about through space is useful as a picture, but leads to absurdities and contradictions if we treat it as a reality. Professor Poynting gave a well-known formula which tells us how energy may be pictured as flowing in a certain way, but the picture is far too artificial to be treated as a reality; for instance if an ordinary bar-magnet is electrified and left standing at rest, the formula pictures energy flowing endlessly round and round the magnet, rather like innumerable rings of children joining hands and dancing to all eternity round a maypole. The mathematician brings the whole problem back to reality by treating this flow of energy as a mere mathematical abstraction. Indeed he is almost compelled to go further and treat energy itself as a mere mathematical abstraction —the constant of integration in a differential equation.

8. The "quantization" of energy.

Whatever the manner in which energy travels, in electro-magnetic radiation, between transmitter and receiver, it is now accepted that it originates, and is absorbed, in amounts of *definite magnitude* which are indivisible. This theory originated

* See the "Epilogue" to this book.

with Max Planck, who postulated it in order to account for certain phenomena in black-body radiation which were inconsistent with the older theory; to-day it is the basis of our knowledge of photo-electric cells, and, together with Einstein's theory of relativity, has produced a complete revolution in theoretical physics. Put briefly, the theory states that the energy of an electro-magnetic wave is lumped in indivisible quanta or "photons", whose energy-content is

$$W = hf, \tag{18b}$$

where f is the frequency of the radiation in cycles per second, and h is a constant (Planck's constant) having the value

$$h = 6{\cdot}625 \times 10^{-27}\,\text{erg-sec.} = 6{\cdot}625 \times 10^{-34}\,\text{joule-sec.}$$

PART II

THE VECTOR POTENTIAL OF THE ELECTRIC CURRENT

1. Definition and function.

In discussing the magnetic field of an electric current, we have stressed the viewpoint that it is merely a successful working hypothesis which is helpful in dealing with the mutual effects of moving charges. However, we are so well acquainted with it that we may be shocked by the suggestion that it is not fundamentally indispensable, and that it may be replaced, in certain circumstances, by a more suitable concept.

In dealing with the induction of an e.m.f. by a changing magnetic field, we make use of Faraday's law of induction in the form

$$e = -\frac{d\phi}{dt} = \oint \mathbf{E} \cdot d\mathbf{l}.$$

Now this equation gives us no information about the value of the induced electric field, E, at any point, but only the value of the line-integral of E around a closed path, so that the usefulness of the magnetic-field concept is limited after all. Further, owing to the same limitation, it gives little help in relating the magnitude of the fields, at any point in an electromagnetic wave, with the particular current disturbance to

which the wave is due. We conclude, then, that though the magnetic field is the best possible hypothesis with which to deal with the mutual effects of electric charges in *uniform* motion (i.e. steady currents), it is not so successful in the case of *accelerating* charges (i.e. changing currents), and that there is room for a new concept, provided it can be designed to tell us what the magnetic field fails to tell.

Such a new concept is provided in the "vector-potential field" of an electric current, and it is specifically designed to

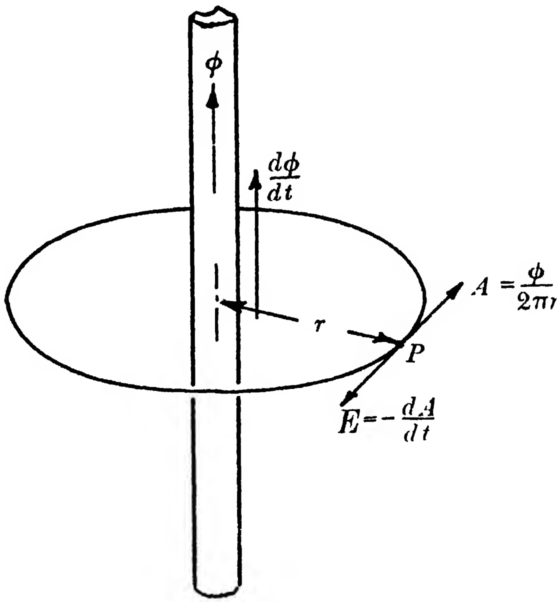


Fig. 126. Vector potential

give the value of the induced electric field (E) at any point on a path surrounding a changing magnetic field. Later we shall see how to connect it directly with the electric current, thus eliminating the magnetic field from our calculations.

Consider a portion of an infinitely long solenoid (i.e. an infinitely large toroid), Fig. 126. The magnetic flux, ϕ, is confined to the space within the coil, and no magnetic field exists at an external point P. Yet when ϕ changes, an e.m.f. $e = -d\phi/dt$ is induced around any closed path surrounding the solenoid. Consider the closed path of a circle of radius r,

passing through P, and coaxial with the solenoid, then it follows from the symmetry of this path that the induced electric field, E, at any point on the path has the same value, and is in the direction of the tangent to the path at the point That is,

$$e=\oint E\,dl=2\pi rE=-\frac{d\phi}{dt}.$$

Now let A be a vector quantity at the point P, whose "line of force" is the coaxial circle of radius r, and whose direction is related to ϕ by the right-handed screw rule, such that

$$\oint A\,dl=\phi\,* \tag{19}$$

Then in the case considered, A is constant around the path and is given by

$$A=\frac{\phi}{2\pi r}\text{ (webers per metre)},$$

so that the induced field around the path is

$$E=-\frac{1}{2\pi r}\frac{d\phi}{dt}=-\frac{dA}{dt}. \tag{20}$$

In the general case, for a non-symmetrical path, we have

$$\oint \mathbf{A}\cdot\mathbf{dl}=\phi,$$

whence, differentiating A with respect to t under the integral sign.

$$\oint\frac{d\mathbf{A}}{dt}\cdot\mathbf{dl}=\frac{d\phi}{dt}=-\oint\mathbf{E}\cdot\mathbf{dl},$$

so that $$\mathbf{E}=-\frac{d\mathbf{A}}{dt}.\text{ as before }\dagger$$

The vector quantity A is called the "vector potential", and if its value at any point is known then the induced electric field at the point, due to changing currents, is given immediately from 5(20).

* This is equivalent to the vector equation Curl $\mathbf{A}=\mathbf{B}$. In addition to this relation, we specify that A does not contain any component obeying the inverse-square law. That is, A is a circuital vector, or div $\mathbf{A}=0$.

† $\mathbf{E}$ does not contain any electro static component, for which of course $\oint dl$ would be zero

Owing to the form of its definition [equation 5(19)], it is usually called the "vector potential of the magnetic field",* or the "magnetic vector potential". This nomenclature is unfortunate, as it gives the impression that A is a property of the magnetic field, whereas in reality it *entirely replaces* the magnetic field, and can be used directly to connect accelerating charges (changing currents) with the electric field they induce, without the intermediate use of the magnetic field. Further, at the point P in Fig. 126, there is no magnetic field whereas there *is* a vector potential field. Consequently we shall call A the "vector potential of the electric current", just as we may call B the "magnetic flux density of the electric current".

2. The relation between A and B in a homogeneous medium.

Let the vector potential, A, at a point P be resolved into components A_x, A_y and A_z parallel to the axes of x, y and z,

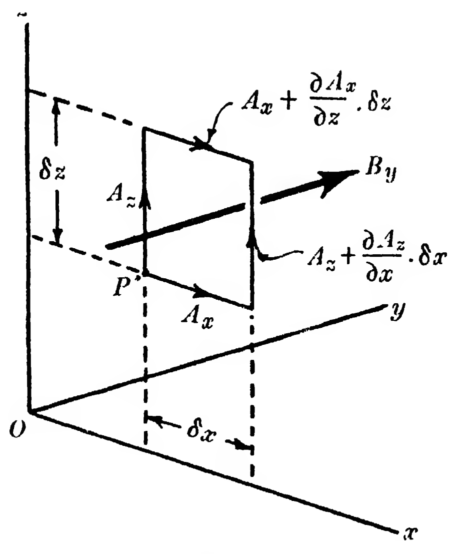


Fig. 127

* First introduced by Franz Neumann in 1845, it was used by Weber, Kirchhoff, Kelvin, and Maxwell. The latter used the name "electrotonic intensity", but later (*Electricity and Magnetism*, vol. II, Article 617) used the name adopted in this book: "The vector potential of the electric current."

and let B_x, B_y and B_z be the corresponding components of the magnetic flux-density at the point.

Apply equation 5(19) to the component B_y of the magnetic field, by taking the line-integral of A around a small rectangle of sides δx and δz in the z-x plane. From Fig. 127 we see that

$$A_z\delta z+\left(A_x+\frac{\partial A_x}{\partial z}\delta z\right)\delta x-\left(A_z+\frac{\partial A_z}{\partial x}\delta x\right)\delta z-A_x\delta x=B_y\delta x\,\delta z,$$

i.e.
$$\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}=B_y. \tag{21}$$

The same method, applied to the components B_z and B_x, gives

$$\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=B_z \tag{22}$$

and
$$\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=B_x. \tag{23}$$

In certain cases it is possible to calculate A at a point near a current system where it is not convenient to calculate B. In such cases the above equations enable B to be found in an indirect manner.

As an illustration of equations 5(21), (22), and (23), consider the infinite solenoid (which may be an infinitely long permanent bar-magnet) of Fig. 126. Take the axis of the solenoid or magnet as the axis of z (Fig. 128) and consider a point $P(x, y)$ in the x-y plane, distant r from this axis.

Then the vector potential at P is perpendicular both to the axis of z and to the radius-vector OP, and is given by

$$A_p=\frac{\phi}{2\pi r}.$$

Its components in the x, y and z directions are (Fig. 128, b)

$$A_x=-A_p\frac{y}{r}=-\frac{\phi}{2\pi}\frac{y}{x^2+y^2},$$

$$A_y=A_p\frac{x}{r}=\frac{\phi}{2\pi}\frac{x}{x^2+y^2},$$

$$A_z=0,$$

whence, from 5(21), (22) and (23):

$$B_x = 0,$$

$$B_y = 0,$$

$$B_z = \frac{\phi}{2\pi}\left[\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right] = 0.$$

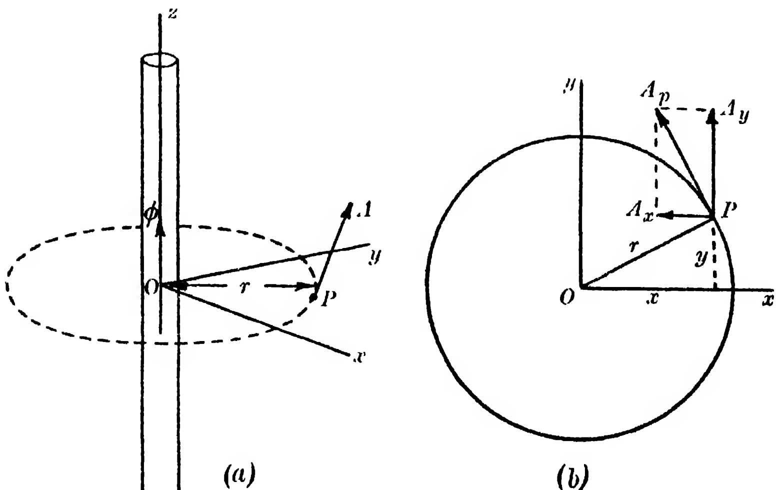


Fig. 128

That is, there is no magnetic field whatever at the point P, and the electric field induced at P when ϕ changes can be considered to be due to the change of the vector potential field at P.

3. The vector potential at a point due to a moving charge or a current element.

Let a charge q be moving with a constant velocity v ($\ll c$). We wish to find the vector potential, A, at any point P in the surrounding medium, due to this moving charge.

Let the medium be homogeneous, of relative permeability μ. Choose axes so that q is at the origin at the instant considered and moving along the axis of x, while the point $P(x, z)$ is in the z-x plane (Fig. 129).

Then the components of the magnetic field at P, due to q, are, from 3(3)

$$B_x = 0,$$
$$B_y = -\frac{\mu\mu_0 qv}{4\pi r^2}\sin\alpha = -\left(\frac{\mu\mu_0 qv}{4\pi}\right)\frac{z}{(x^2+z^2)^{3/2}},$$
$$B_z = 0.$$

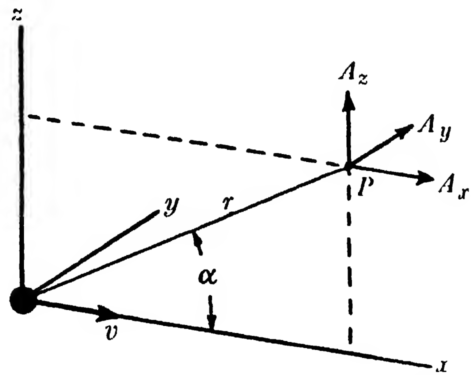


Fig. 129. Vector potential of moving charge

Substituting in 5(21), (22) and (23):

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\left(\frac{\mu\mu_0 qv}{4\pi}\right)\frac{z}{(x^2+z^2)^{3/2}},$$
$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0,$$
$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0.$$

It is readily seen that a solution of these equations is:

$$\left.\begin{aligned} A_x &= \frac{\mu\mu_0 qv}{4\pi}\,\frac{1}{(x^2+z^2)^{1/2}} = \frac{\mu\mu_0}{4\pi}\left(\frac{qv}{r}\right),^* \\ A_y &= 0, \\ A_z &= 0. \end{aligned}\right\} \qquad (24)$$

Thus the vector potential at P is *parallel* to the direction of motion of the charge, and inversely proportional to the distance from P to the charge. This relation gave rise to the

* This is the simplest solution. For an alternative, due to W. Weber, see E. G. Cullwick, *Electromagnetism and Relativity*, p. 218.

name "vector potential", owing to the analogous form of the expression for the electric potential at a point due to a charge

$$U=\frac{q}{4\pi\epsilon_0 r},$$

which is a *scalar* quantity.

For a current element, we put $I\,\delta l=qv$, so that the value of A at a point due to a current I flowing in a short path δl is

$$\mathbf{A}=\mu\mu_0\frac{I\,\boldsymbol{\delta}\mathbf{l}}{4\pi r}, \qquad (25)$$

which should be compared with the value of the magnetic field of the same element:

$$B=\mu B_0=\mu\mu_0\frac{I\,\delta l}{4\pi r^2}\sin\alpha. \qquad \text{from } 3(4)$$

Both these equations, of course, are true only for the particular cases where $B=\mu B_0$.

The directions of the vectors A and B, due to the current element, are shown graphically in Fig. 130.

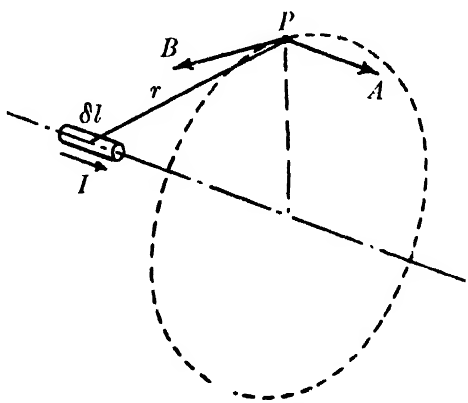


Fig. 130. Vector potential and magnetic field of current element. A is parallel to i, and B is related to i by the right-handed screw rule.

The vector potential of a flux-filament. It may sometimes be desirable to calculate the vector potential at a point, due to currents whose circuits are too complex for the use of 5(25), but whose magnetic field has a simple configuration. For

example, if the magnetic field is that in an iron core, the approximate configuration of the field may be known whereas 5(25) cannot be applied to the equivalent current-circuits of the individual iron atoms in the core.

Now since
$$\oint \mathbf{A} \cdot \mathbf{dl} = \phi \qquad 5(19)$$
and
$$\oint \mathbf{H} \cdot \mathbf{dl} = I,$$
while
$$H = \frac{I\,\delta l}{4\pi r^2} \sin\alpha, \qquad 3(33)$$

it follows that, if ϕ is the flux in a thin filament of length δl and area δS, whose sides are coincident with the lines of flux, the vector potential due to this flux-filament, at a point distant r from it, is
$$A = \frac{\phi\,\delta l}{4\pi r^2} \sin\alpha, \qquad (25a)$$

where α has the same meaning as in 3(33), and
$$\phi = B\,\delta S.$$

Since the whole of the flux in a given magnetic system may be divided up into such flux-filaments, it follows that the vector potential at a point, due to the system, may be obtained from the expression
$$A = \iiint \frac{B \sin\alpha}{4\pi r^2}\, dl\, dS, \qquad (25b)$$

where the integration is to be performed over the whole of the space occupied by the field.

In certain cases of simple flux-configuration this method of calculation may be useful, and as an example the reader should refer to Ex. 14 of this chapter.

4. The electric field at a point induced by transformer and motional action.

We have already seen, in Chapters II and III, that a source of a magnetic field, when moving in a straight line with velocity v, induces an electric field whose intensity is given by
$$\mathbf{E} = \mathbf{B} \times \mathbf{v}. \qquad 2(12b)$$

This law is of course complementary to the law for the force experienced by a unit charge moving with velocity $\mathbf{v}$ in a magnetic field of flux-density $\mathbf{B}$, i.e. $\mathbf{F} = \mathbf{v} \times \mathbf{B}$, and is just as fundamental. It is important, therefore, to see if any more detailed analysis of this law is possible.

The first point to be noted is that the circuital law of the electric field,

$$e = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt}, \qquad 3(26)$$

is valid no matter how the change of linking flux is brought about. The flux-change may be due to a stationary current-circuit or electromagnet whose magnetic field is changing, so that the induced e.m.f. is entirely "transformer", or $d\phi/dt$ may be due to the motion of a constant-field circuit or magnet so that the e.m.f. is entirely "motional", or we may have a combination of both effects. It follows, therefore, that the three equations 5(4), 5(5) and 5(6) which are the three-dimensional form of the vector equation $\text{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$, are generally valid and not limited to transformer induction alone. If these equations are compared with 5(21)–5(23), which are equivalent to the vector equation $\text{curl}\,\mathbf{A} = \mathbf{B}$, it is seen at once that the two sets of equations are equivalent if

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \qquad 5(26)^*$$

so that we may accept 5(26) as a general expression for the electric field induced at a point by a changing magnetic field.

It should be noted, however, that the electric field $\mathbf{E}$ in equation 3(26) may contain an electrostatic component, $\mathbf{E}_s$ say, due to charge distributions and which can be expressed as the negative gradient of the electrostatic potential U. Such a component does not contribute to the e.m.f in the closed circuit and is therefore not included in the value of $\mathbf{E}$ given

* When $\mathbf{A}$ is due to a moving source, $\partial\mathbf{A}/\partial t$ means the time-rate of change of $\mathbf{A}$ at a point fixed in the reference system in which the source is moving, while $d\mathbf{A}/dt$ denotes the time rate of change of $\mathbf{A}$ at a point moving with the source. If $\mathbf{v}$ is the velocity of the source,

$$\partial\mathbf{A}/\partial t = d\mathbf{A}/dt - (\mathbf{v}\,.\,\nabla)\,\mathbf{A}.$$

by 5(26). The *total* electric field at a point must therefore be

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} U, \qquad (26a)$$

the first term giving the component of field induced by a changing magnetic field and the second term the electrostatic component due to the distribution of charges Equation 5(26a) is the general expression for the electric field in terms of the scalar and vector potentials.

It necessarily follows that the motional electric field of a moving current-circuit or magnet, $\mathbf{E} = \mathbf{B} \times \mathbf{v}$, can be expressed in the form of equation 5(26a), and it may be shown by a simple application of vector analysis that

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} = -\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}(\mathbf{v} \cdot \mathbf{A}) \qquad (26b)^*$$

The motional electric field therefore consists, in general, of a *solenoidal* component $-\partial \mathbf{A}/\partial t$ which fully accounts for the induced e m.f. in a stationary closed circuit, and a *polar* part, $-\operatorname{grad}(\mathbf{v} \cdot \mathbf{A})$, which must be attributed to a charge distribution, having a scalar potential $\mathbf{v} \cdot \mathbf{A}$, on the moving field-source.

If the m.m.f. of the moving field-source is changing, there will be a transformer component of electric field in addition to the motional field Provided the velocity of the source is small compared with c, the transformer component will be independent of the velocity and will be given by

$$\mathbf{E}_t = -\frac{d\mathbf{A}}{dt}, \qquad (26c)$$

where $d\mathbf{A}/dt$ has the meaning given in the footnote on p. 277. The time-rate of change of $\mathbf{A}$ at a fixed point, $\partial \mathbf{A}/\partial t$, will now include both $d\mathbf{A}/dt$ and the rate of change of $\mathbf{A}$ due to the motion, and the resultant electric field can be expressed in either of the two forms

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad}(\mathbf{v} \cdot \mathbf{A}) \qquad (26d)$$

and

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} - \frac{d\mathbf{A}}{dt}. \qquad (26e)$$

* See E. G. Cullwick, *Electromagnetism and Relativity*, pp. 47–48.

In 5(26d) the total field is divided into solenoidal and polar components, while in 5(26e) it is divided into motional and transformer fields.

In Chapter III, Section 7, it was shown that the use of the relativistic law for the transformation of velocity leads to the conclusion that a straight current-carrying conductor, moving longitudinally, carries a static charge proportional to the velocity. This is the explanation of the polar component $-\text{grad}\,(\mathbf{v}\cdot\mathbf{A})$ of the motional electric field, and applies equally to the case of a moving magnet since this can be considered to consist of microscopic current loops.

A particular case in which there is no polar component is that of a current-element (or straight wire) moving perpendicularly to its length.

Consider a short element of a conductor, δl, parallel to the axis of z, carrying a changing current i, and situated at the point (x, z), Fig. 131. Take the origin of co-ordinates at the point where the induced electric field is to be found, and let the element be moving with velocity v along the axis of x.

For simplicity assume that the medium is non-magnetic. Then the vector potential at O due to $i\,\delta l$ is

$$A_x = A_y = 0,$$

and

$$A_z = \mu_0 \frac{i\,\delta l}{4\pi r} = \frac{\mu_0 i\,\delta l}{4\pi(x^2+z^2)^{\frac{1}{2}}}.$$

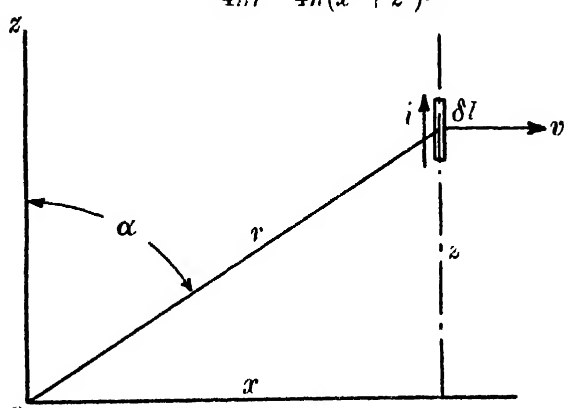


Fig. 131. Moving current element

The induced electric field, given by $E = -dA/dt$, is

$$E_x = E_y = 0,$$

and
$$E_z = -\frac{dA_z}{dt} = \mu_0 \frac{\delta l}{4\pi r}\frac{di}{dt} + \frac{\mu_0 ix\,\delta l}{4\pi(x^2+z^2)^{\frac{3}{2}}}\frac{dx}{\delta t},$$

but
$$\frac{dx}{dt} = v,$$

and
$$\frac{\mu_0 ix\,\delta l}{4\pi(x^2+z^2)^{\frac{3}{2}}} = \mu_0 \frac{i\,\delta l}{4\pi r^2}\sin\alpha = B,$$

the magnetic field at O due to the element.

Thus by taking the complete differential coefficient of A with respect to t we obtain

$$E_z = -\frac{\mu_0}{4\pi}\left(\frac{\delta l}{r}\right)\frac{di}{dt} + Bv.$$

The first term, being independent of the motion, is the transformer field component, corresponding to $-d\mathbf{A}/dt$ in 5(26e). The second term, Bv, is the motional field corresponding to $\mathbf{B}\times\mathbf{v}$ in 5(26e) and contains no polar part because $\mathbf{v}.\mathbf{A}$ is zero.

The transformer field of a stationary circuit. We may now make use of 5(25), the vector potential of a current element, to obtain an expression for the electric field induced by a changing current in a complete stationary circuit, provided di/dt has the same value at all points of the circuit It is clearly given by

$$\mathbf{E} = -\frac{\mu_0}{4\pi}\oint\frac{di}{dt}\frac{dl}{r}. \tag{27}$$

where the integration is performed around the whole of the primary circuit, and dl is a vector.

This result is true for a closed circuit. From it we may postulate that an accelerating charge acts on a second charge with a force given by

$$\mathbf{F} = -\frac{\mu_0}{4\pi}\left(\frac{q_1 q_2}{r}\right)\frac{d\mathbf{v}}{dt} \quad \text{see (Fig. 132)}, \tag{28}$$

and that the electric field induced at a point distant r from a charge q moving with acceleration α is

$$\mathbf{E} = -\frac{\mu_0}{4\pi}\frac{q\alpha}{r}. \tag{28a}$$

5. Neumann's expression for inductance.

A. *Mutual inductance.* Consider two neighbouring circuits, (1) and (2), Fig. 133. If the current i_1 in (1) changes, an e.m.f. is induced in (2):

$$e_2 = \oint \mathbf{E}_2 \cdot d\mathbf{l}_2 = -M\frac{di_1}{dt},$$

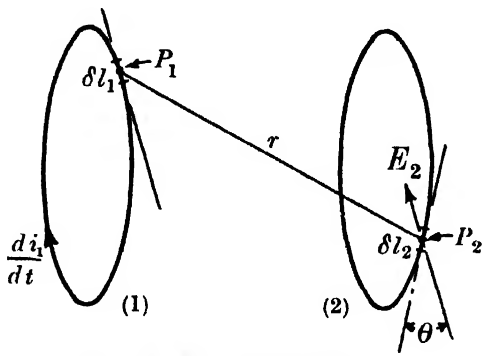


Fig. 133. Mutual inductance

where E_2 is the electric field at P_2 induced by di_1/dt, and θ is the angle between E_2 and δl_2.

But, from 5(27),

$$\mathbf{E}_2 = -\frac{\mu_0}{4\pi}\oint \frac{di_1}{dt}\frac{d\mathbf{l}_1}{r},$$

where the integration is performed around the complete circuit of (1), and r is the distance between δl_1 and δl_2.

Whence, provided that di_1/dt has the same value at all points of circuit (1),

$$M = \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r}, \qquad (29)$$

where θ is the angle between δl_1 and δl_2.

B. *Self-inductance.* For the self-inductance of a circuit equation 5(29) still holds, but both δl_1 and δl_2 must now be taken on the same circuit. The calculation in a practical case is not so convenient since the dimensions of the conductor, and the distribution of the current over its section, must be considered.

Example. The mutual inductance of two parallel wires. One of the simpler applications of Neumann's Integral [5(29)] is in

the calculation of the mutual inductance of two fine parallel wires of length L and distant D apart (Fig. 134).

Consider an element δl of the wire AB, distant l from A, and an element δx of the wire CD, distant x from C. Then the mutual inductance of these two elements is given by

$$\delta' M = \frac{\mu_0}{4\pi}\frac{\delta x\,\delta l}{r} = \frac{\mu_0}{4\pi}\frac{\delta x\,\delta l}{\sqrt{(x-l)^2+D^2}},$$

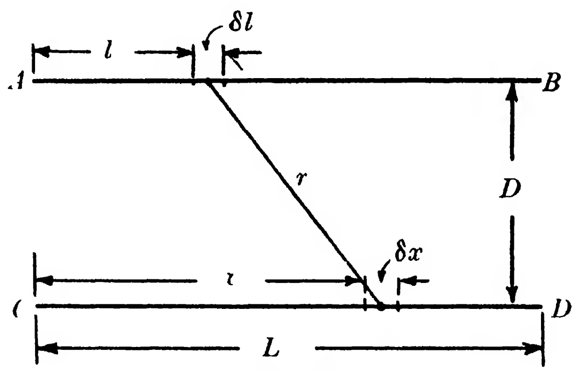


Fig 134

so that the mutual inductance of δx and the whole length of AB is

$$\delta M = \int_{l=0}^{l=L} d'M = -\frac{\mu_0}{4\pi}(\delta x)\left[\log_e \frac{(x-l)+\sqrt{(x-l)^2+D^2}}{D}\right]_{l=0}^{l=L}$$

$$= -\frac{\mu_0}{4\pi}(\delta x)\left[\log_e \frac{(x-L)+\sqrt{(x-L)^2+D^2}}{D} - \log_e \frac{x+\sqrt{x^2+D^2}}{D}\right],$$

and the total mutual inductance of AB and CD is

$$M = \int_{x=0}^{x=L} dM.$$

Now

$$\int \log_e \frac{x+\sqrt{x^2+D^2}}{D}\,dx = x\log_e \frac{x+\sqrt{x^2+D^2}}{D} - \sqrt{x^2+D^2},$$

so that

$$M = \frac{\mu_0}{2\pi}\left[L\log_e \frac{L+\sqrt{L^2+D^2}}{D} - \sqrt{L^2+D^2}+D\right] \qquad (30)$$

$$\left[\text{since}\left(\frac{L+\sqrt{L^2+D^2}}{D}\right)\left(\frac{-L+\sqrt{L^2+D^2}}{D}\right)=1\right].$$

6. Retarded potentials.

The reader has probably had an uncomfortable feeling that equation 5(27) is an incomplete statement of the electric field induced by a changing current, since in deriving it we have neglected the effect of the displacement currents of this induced field. This is perfectly true, and our sole justification lies in the fact that, when the Maxwell hypothesis of displacement current is included, equations arise which show that the vector potential of a current element, at a point, is still given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\left(\frac{i\,\delta\mathbf{l}}{r}\right), \tag{5(25)}$$

and the electric field induced when the current changes in a complete circuit, at a point fixed relatively to the circuit, by

$$\mathbf{E} = -\frac{d\mathbf{A}}{dt} = -\frac{\mu_0}{4\pi}\oint\frac{di}{dt}\frac{d\mathbf{l}}{r}, \tag{5(27)}$$

provided that, if A and E are to be calculated at an instant t, then i must be the instantaneous value of the current in a fixed element δl (distant r from the point) at an instant $(t-r/c)$.*

The same equations show that this "retardation" or time-lag effect must also be included in calculating the electrostatic potential, U, at a point due to the instantaneous *position* of charges, from which the electro-static component of the electric field is obtained from the relation

$$E_s = -\frac{\partial U}{\partial l}.$$

That is, U (at time t) is to be calculated by the equation

$$U = \frac{q}{4\pi\epsilon_0 r}, \tag{1(8)}$$

where q is to be taken as the charge at a distance r from the point at the instant $(t-r/c)$.

Quantities calculated in this way are called "retarded

* See, for example, E. B. Moullin, *Principles of Electromagnetism*, pp. 262–5. Also M. Mason, *Phys. Rev.* xv (1920), p. 312.

functions" (in this case they are "retarded potentials") and to remind us of this fact the above equations are written

$$\mathbf{A} = \mu_0 \frac{[i]\,\delta\mathbf{l}}{4\pi r} \tag{31}$$

and

$$U = \frac{[q]}{4\pi\epsilon_0 r}, \tag{32}$$

the square brackets indicating the quantities whose values must be those at the earlier instant.

These equations are applicable only if the current-element, compared with r, has a small diameter and if q is a point charge or has spherical symmetry. In general, if in a volume element $\delta\tau$ of a conductor there is a current density $\mathbf{J}$ and a charge density ρ, the equations become

$$\mathbf{A} = \frac{\mu_0}{4\pi}\iiint \frac{[\mathbf{J}]}{r}\,d\tau, \quad U = \frac{1}{4\pi\epsilon_0}\iiint \frac{[\rho]}{r}\,d\tau \tag{33}$$

The practical uses of the vector potential. Although the vector potential is ideal in the calculation of the induced electric field at a point due to changing currents, it does not follow that it is the most convenient concept to use in the theory of iron-cored electrical machines. Indeed, the magnetic flux, ϕ, is far too useful to be replaced in the theory of dynamos and transformers. The field-concepts should be regarded as vector-tools of equal status and value, the most convenient being used in any particular case.

The vector potential, as pointed out by Maxwell,* is the mathematical equivalent of Faraday's "electrotonic state". Faraday evidently felt this to be fundamental to the electromagnetic induction of currents, but he sacrificed the idea to the lines of force of the magnetic field, a concept already familiar in terms of permanent magnets. Later in his life, however, he stated: "Again and again the idea of an electrotonic state has been forced upon my mind."† As Maxwell wrote:

* *Treatise*, II, Article 540.

† *Experimental Researches*, Series II, 3269.

"The scientific value of Faraday's conception of an electrotonic state consists in its directing the mind to lay hold of a certain quantity, on the changes of which the actual phenomena depend."*

The practical applications of the vector potential, in modern electrical theory, are usually limited to two:

(1) It is useful in determining the configuration of the magnetic fields of steady-current systems, such as those of conductors of finite sections of various shapes, and of conductors in the neighbourhood of iron (in slots, etc.) of given configuration. In such cases it is sometimes possible to calculate the value of A at a point, and the components of the magnetic flux-density are then determined by means of equations 5(21), (22) and (23). In such work, since the currents are steady, the "retardation" of the potential is of no consequence.†

(2) It is used in the calculation of the electric field induced at a point by a high-frequency current in a circuit, such as a transmitting radio antenna, and the energy radiated in the "electro-magnetic waves" of such a current. In this work the retardations of the potentials are of prime importance. One of the simpler examples of this application is given in the next section.‡

In the latter application, the three-co-ordinate components of the total electric field at a point which is stationary with respect to the radiating circuit may be expressed as follows.

Let U be the electro-static (scaler) potential, due to the distribution of charges on the circuit, at the point. Then

$$E_s = -\frac{\partial U}{\partial l}, \qquad 1(9)$$

so that the components of E_s in the directions of the x, y and z axes are

$$E_{sx} = -\frac{\partial U}{\partial x},$$

$$E_{sy} = -\frac{\partial U}{\partial y},$$

$$E_{sz} = -\frac{\partial U}{\partial z},$$

and, if A_x, A_y and A_z are the components of the vector potential, A, at the point, the three components of the total electric field are

$$\left.\begin{aligned} E_x &= -\frac{\partial U}{\partial x} - \frac{\partial A_x}{\partial t}, \\ E_y &= -\frac{\partial U}{\partial y} - \frac{\partial A_y}{\partial t}, \\ E_z &= -\frac{\partial U}{\partial z} - \frac{\partial A_z}{\partial t}, \end{aligned}\right\} \qquad (34)$$

where A and U are to be calculated from equations 5(31) and 5(32) or 5(33).

7. The energy radiated by a high-frequency current in a straight wire terminated by conducting spheres: radiation resistance.

The calculation of the energy radiated from a practical aerial is difficult because the exact distribution of the conduction current at any instant is not known, but in order to illustrate the use of the above equation, 5(34), we may take the case of an idealized aerial, the current and charge distribution in which may be postulated to a fair degree of accuracy.

Fig. 135 represents an isolated straight wire of length L, terminated by two conducting spheres whose radius is large compared with that of the wire, but small compared with L.

At any instant t, let the current in the wire be i and the charges on the spheres $+q$ and $-q$. Let $q = Q\cos\omega t$, then if we assume that there is no net charge on any portion of the

surface of the wire, the current i will have the same value at all points of the wire and will be given by

$$i = \frac{dq}{dt} = -\omega Q \sin \omega t = I_m \sin \omega t,$$

where $I_m = -\omega Q$.

(This simplified distribution of charge and current is justifiable provided that the wire is very thin and its length small compared with the wave-length $\lambda = 2\pi c/\omega$.)

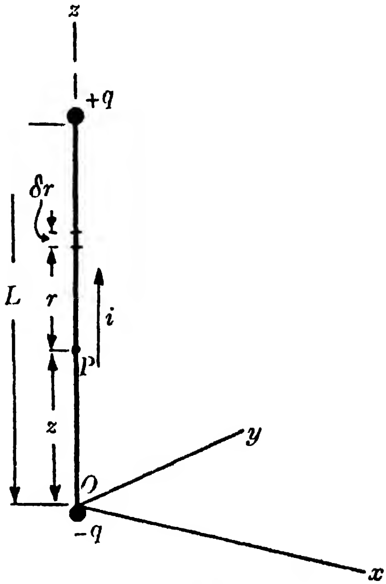


Fig. 135. Dipole antenna

The problem is to find the mean rate at which energy must be supplied to the system in order to maintain this current. If we neglect the resistance of the wire, the whole of this energy will be radiated into the surrounding space. It is assumed that the reader is familiar with the following facts of elementary alternating-current theory.

If a current $i = I_m \sin \omega t$ flows in a circuit of inductance L and effective resistance R_e, the e.m.f. required to maintain the current is

$$e = I_m \omega L \cos \omega t + I_m R_e \sin \omega t.$$

The first term is in *phase quadrature* with the current, and involves no average absorption of energy, since the energy taken in during one quarter-cycle is released during the next quarter-cycle. The second term, however, represents a voltage component *in-phase* with the current, and involves an absorption of energy at the average rate of

$$P = \frac{I_m^2 R_e}{2}. \tag{35}$$

In the present case, then, the problem is to find an expression for the *in-phase component* of the voltage between the ends of

the wire, and to do this we first find the field intensity (E_z) at any point P of the wire, whose axis is taken coincident with the axis of z.

From 5(34):
$$E_z = -\frac{\partial U}{\partial z} - \frac{\partial A_z}{\partial t},$$

where A_z is the vector potential at P due to the current in the wire, and U is the electric potential due to the net charges on the terminal spheres.

First consider the potential U. At the instant t, its value at P, due to the lower charge $-q$ alone, is

$$U_1 = \frac{-q}{4\pi\epsilon_0 z},$$

where $-q = -Q\cos\omega(t-z/c)$, since the potential is retarded, i.e.

$$U_1 = -\frac{Q}{4\pi\kappa_0 z}\left(\cos\frac{\omega z}{c}\cos\omega t + \sin\frac{\omega z}{c}\sin\omega t\right).$$

Only the second term will give rise to a component of E_z in phase with the current. Now since

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \text{etc.},$$

we can write this in-phase component:

$$U_1 \doteqdot -\frac{Q}{4\pi\epsilon_0}\left(\frac{\omega}{c} - \frac{\omega^3 z^2}{6c^3} + \frac{\omega^5 z^4}{120c^5}\right)\sin\omega t.$$

Similarly, the in-phase component of U due to the charge $+q$ on the upper sphere is

$$U_2 \doteqdot +\frac{Q}{4\pi\epsilon_0}\left(\frac{\omega}{c} - \frac{\omega^3(L-z)^2}{6c^3} + \frac{\omega^5(L-z)^4}{120c^5}\right)\sin\omega t,$$

so that the total electric potential at P is

$$U = U_1 + U_2 \doteqdot -\frac{Q}{4\pi\epsilon_0}\frac{\omega^3 L(L-2z)}{6c^3}\left[1 - \frac{\omega^2}{20c^2}(L^2 - 2Lz + 2z^2)\right]\sin\omega t.$$

The in-phase component of E_z due to U is

$$E_s = -\frac{\partial U}{\partial z} = -\frac{Q}{4\pi\epsilon_0}\frac{\omega^3 L}{3c^3}\left[1 - \frac{\omega^2}{10c^2}(L^2 - 3Lz + 3z^2)\right]\sin\omega t.$$

Next, consider the component of E_z due to the changing current

in the wire. The vector potential at P due to a current element $i\,\delta r$ distant r from P is

$$A_z = \mu_0 \frac{[i]\,\delta r}{4\pi r} = \mu_0 \frac{I_m \sin\omega\left(t - \frac{r}{c}\right)\delta r}{4\pi r}$$

$$= -\mu_0 \frac{\omega Q \sin\omega\left(t - \frac{r}{c}\right)\delta r}{4\pi r}$$

$$= -\frac{\mu_0\,\omega Q\,\delta r}{4\pi r}\left(\cos\frac{\omega r}{c}\sin\omega t - \sin\frac{\omega r}{c}\cos\omega t\right).$$

The induced electric field at P due to the changing current in δr is

$$E = -\frac{\partial A_z}{\partial t} = \frac{\mu_0 Q\omega^2\,\delta r}{4\pi r}\left(\cos\frac{\omega r}{c}\cos\omega t + \sin\frac{\omega r}{c}\sin\omega t\right).$$

Again, only the second term contributes to the in-phase component of E. This becomes, on expanding $\sin\omega r/c$,

$$E' = \frac{\mu_0 Q\omega^3\,\delta r}{4\pi c}\left(1 - \frac{\omega^2 r^2}{6c^2} + \ldots\right)\sin\omega t.$$

The electric field at P due to the changing current in the whole wire is then

$$E_a = \int_z^{L-z} E'\,dr \doteqdot \frac{Q\omega^3 L}{4\pi\epsilon_0 c^3}\left[1 - \frac{\omega^2}{18c^2}(L^2 - 3Lz + 3z^2)\right]\sin\omega t,$$

since

$$\mu_0 = \frac{1}{\epsilon_0 c^2}.$$

The total electric field at P due to q, $-q$ and i is then

$$E_z = E_s + E_a \doteqdot \frac{Q\omega^3 L}{4\pi\epsilon_0 c^3}\left[\frac{2}{3} - \frac{1}{45}\frac{c^2}{c^2}(L^2 - 3Lz + 3z^2)\right]\sin\omega t.$$

Now since we have agreed that $\omega L \ll c$, we may neglect the second term, so that

$$E_z \doteqdot \frac{Q\omega^3 L}{6\pi\epsilon_0 c^3}\sin\omega t = -\frac{\omega^2 L}{6\pi\epsilon_0 c^3} I_m \sin\omega t,$$

which is independent of the position of P on the wire.

The in-phase component of the e.m.f. along the wire is therefore

$$e \doteqdot E_z L = -\frac{\omega^2 L^2}{6\pi\epsilon_0 c^3} I_m \sin\omega t = E_m \sin\omega t,$$

opposing the flow of current.

In order to maintain the current flow, an in-phase component of voltage, $-e$, must be maintained by a generator, and the average rate at which energy must be supplied (i.e. the average power) is

$$P = \frac{(-E_m)\,I_m}{2} = I_m^2\left(\frac{\omega^2 L^2}{12\pi\epsilon_0 c^3}\right) = \mu_0 I_m^2\left(\frac{\omega^2 L^2}{12\pi c}\right). \quad (36)$$

Comparing this expression with 5(35), the effective resistance (or "radiation resistance") is seen to be

$$R_e \doteqdot \frac{\omega^2 L^2}{6\pi\epsilon_0 c^3} = \mu_0\left(\frac{\omega^2 L^2}{6\pi c}\right), \quad (37)$$

or, putting $\omega = 2\pi f$,

$$R_e \doteqdot \frac{2\pi f^2 L^2}{3}\frac{\mu_0}{c} \doteqdot 8{\cdot}8 \times 10^{-15} f^2 L^2 \text{ ohms} \doteqdot 790\frac{L^2}{\lambda^2} \text{ ohms} \quad (38)$$

where f is in cycles per second and L and λ are in metres.

Thus the radiated power is proportional to the square of the frequency.

The quadrature component of the e.m.f., which we have not calculated, gives rise to expressions for the inductance and capacitance of the system. For a complete solution, the work of E. B. Moullin, referred to above, should be consulted.

9. The electric and magnetic fields at a great distance from the same aerial. Use of Poynting's Theorem to calculate the radiated power.

Consider a point P, in the z-x plane, at a great distance r from the upper end, and $r + \delta r$ from the lower end, of the

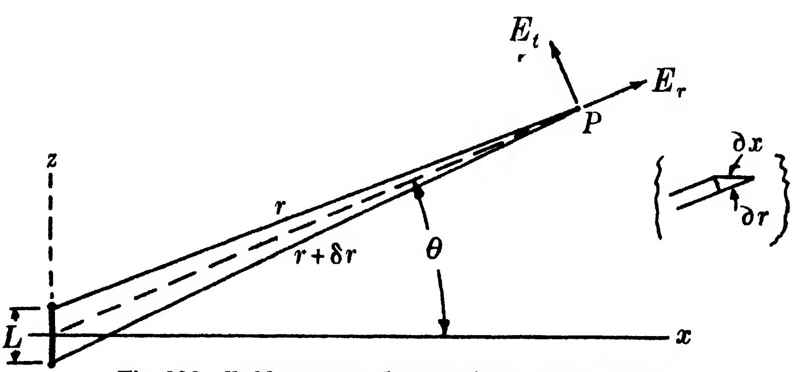


Fig. 136. Fields at great distance from dipole antenna

aerial. Let the line joining P to the centre of L make an angle θ with the axis of x. Then the electric potential at P due to $+q$ and $-q$ is

$$U = \frac{Q}{4\pi\epsilon_0 r}\cos\omega\left(t-\frac{r}{c}\right) - \frac{Q}{4\pi\epsilon_0(r+\delta r)}\cos\omega\left(t-\frac{r+\delta r}{c}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\cos\omega\left(t-\frac{r}{c}\right)\left\{\frac{1}{r}-\frac{1}{r+\delta r}\cos\frac{\omega\,\delta r}{c}\right\}$$

$$-\frac{Q}{4\pi\epsilon_0(r+\delta r)}\sin\frac{\omega\,\delta r}{c}\sin\omega\left(t-\frac{r}{c}\right)$$

$$\doteqdot \frac{Q\,\delta r}{4\pi\epsilon_0 r^2}\cos\omega\left(t-\frac{r}{c}\right) - \frac{Q\omega\,\delta r}{4\pi\epsilon_0 cr}\sin\omega\left(t-\frac{r}{c}\right),$$

since $\omega\,\delta r \ll c$.

But $\delta r = L\sin\theta$, so that

$$U \doteqdot \frac{QL\sin\theta}{4\pi\epsilon_0}\left\{\frac{1}{r^2}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{cr}\sin\omega\left(t-\frac{r}{c}\right)\right\}.$$

The vector potential at P is

$$A_x = A_y = 0,$$

$$A_z = \frac{\mu_0}{4\pi}\int_{z_0}^{z_0+L}\frac{[i]\,dz}{r} = \frac{\mu_0 L}{4\pi r} I_m \sin\omega\left(t-\frac{r}{c}\right)$$

$$= -\frac{\mu_0}{4\pi}\frac{\omega LQ}{r}\sin\omega\left(t-\frac{r}{c}\right).$$

The radial component of E at P is

$$E_r = -\frac{\partial A_z}{\partial t}\sin\theta - \frac{\partial U}{\partial r}$$

$$= \frac{QL}{2\pi\epsilon_0}\sin\theta\left\{\frac{1}{r^3}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{cr^2}\sin\omega\left(t-\frac{r}{c}\right)\right\}$$

And the tangential component of E is

$$E_t = -\frac{\partial A_z}{\partial t}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial\theta}$$

$$= -\frac{QL\cos\theta}{4\pi\epsilon_0}\left\{\frac{1}{r^3}\cos\omega\left(t-\frac{r}{c}\right) - \frac{\omega}{cr^2}\sin\omega\left(t-\frac{r}{c}\right)\right.$$

$$\left. - \frac{\omega^2}{c^2 r}\cos\omega\left(t-\frac{r}{c}\right)\right\}$$

At very great distances from the aerial, the terms containing $1/r^2$ and $1/r^3$ are quite negligible compared with the last term of E_t, so that

$$E_r \doteqdot 0,$$

$$E_t \doteqdot -\frac{QL\omega^2 \cos\theta}{4\pi\epsilon_0 c^2 r} \cos\omega\left(t-\frac{r}{c}\right)$$

$$= -\frac{\mu_0 QL\omega^2 \cos\theta}{4\pi r} \cos\omega\left(t-\frac{r}{c}\right)$$

$$\doteqdot -\frac{\mu_0 I_m \omega L \cos\theta}{4\pi r} \cos\omega\left(t-\frac{r}{c}\right). \qquad (39)$$

The magnetic field at P, if required, is given by 5(21), (22) and (23), where

$$A_x = A_y = 0,$$

$$A_z = \frac{\mu_0 LI_m}{4\pi r} \sin\omega\left(t-\frac{r}{c}\right);$$

whence $\quad B_x = B_z = 0$

and $\quad B_y = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}$

$$= \frac{\mu_0}{4\pi} LI_m \left[\frac{1}{r^2} \sin\omega\left(t-\frac{r}{c}\right) + \frac{\omega}{cr} \cos\omega\left(t-\frac{r}{c}\right)\right] \frac{\partial r}{\partial x};$$

but $\quad \dfrac{\partial r}{\partial x} = \cos\theta \quad$ (Fig 136),

so that, neglecting the term in $1/r^2$,

$$B_y = \frac{\mu_0}{4\pi} \frac{I_m \omega L}{cr} \cos\theta \cos\omega\left(t-\frac{r}{c}\right)$$

$$= -\frac{E_t}{c}. \qquad (40)$$

That is, the relation between E, B and the direction of propagation is that given by 5(12). Neglecting terms in $1/r^2$, etc. is tantamount to saying that the only fields at P are those of the electro-magnetic wave.

The calculation of the radiated power by means of Poynting's Theorem. Having obtained the values of E and B at the point P (at a great distance from the dipole), let us apply Poynting's

Theorem to the surface of a sphere of radius r, having its centre at the dipole. Then the rate of energy flow, through unit area, at P is

$$p = EH, \tag{5(18)}$$

and is radially outwards from the centre of the sphere.

The mean value of p over a complete cycle is

$$p_{\text{mean}} = \frac{E_{\max} H_{\max}}{2}$$

(since E and B are in-phase, and vary sinusoidally with time), where

$$E_{\max} = \frac{\mu_0 I_m \omega L \cos\theta}{4\pi r} \quad \text{from 5(39)}$$

and $$H_{\max} = \frac{B_{\max}}{\mu_0} = \frac{E_{\max}}{\mu_0 c} = \frac{I_m \omega L \cos\theta}{4\pi c r} \quad \text{from 5(40)},$$

so that $$p_{\text{mean}} = \frac{\mu_0 I_m^2 \omega^2 L^2 \cos^2\theta}{32\pi^2 c r^2}.$$

The rate of energy flow through an elementary ring, coaxial with the dipole aerial, of width $r\,\delta\theta$ and radius $r\cos\theta$, is

$$\delta P = (2\pi r^2 \cos\theta\, \delta\theta)\, p,$$

so that the rate of energy flow through the complete surface of the sphere is

$$P = 2\int_{\theta=0}^{\theta=\pi/2} dP = \frac{\mu_0 I_m^2 \omega^2 L^2}{8\pi c}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \mu_0 I_m^2 \left(\frac{\omega^2 L^2}{12\pi c}\right) \quad \text{[see 5(36)]},$$

which is the same result as that obtained by the method of Section 8.

The essential difference between the two methods should be noted. In the first, we concentrate upon conditions in the aerial itself, and calculate the e.m.f. necessary to maintain a given high-frequency current. In the second, we apply Poynting's Theorem to calculate the rate of energy flow through a sphere of very large radius. Both methods give the same result since they are both based upon theories which are consistent with experiment, and which can be converted one into the other.

In the simple and idealized case considered in these two sections, the Poynting method is the simpler, but does not give us the same insight into conditions in the aerial itself as does the first method. For a further comparison of the two methods the second paper referred to on p. 285, by E. B. Moullin, should be consulted.

EXAMPLES, CHAPTER V

1. A vertically polarized electro-magnetic wave has a wave-length of 360 metres. The amplitude of its electric field is 0·01 volt per metre. Find (a) the frequency, (b) the amplitude of the magnetic field component, (c) the amplitude of the e.m.f. induced in a vertical single wire of height 50 metres, (d) the amplitude of the e.m.f. induced in a rectangular loop whose vertical sides are 10 metres long and whose horizontal sides are 60 metres long, its plane being parallel to the direction of propagation, and (e) the mean value of the energy carried, per cubic metre. Assume that the medium is free space.

Solution. (a) From 5(15):

$$f = \frac{c}{\lambda} = \frac{3\times 10^8}{360} = 8{\cdot}34\times 10^5 = 834 \text{ kilo-cycles per sec.}$$

(b) From 5(12):

$$B = \frac{E}{c} = \frac{0{\cdot}01}{3\times 10^8} = 3{\cdot}33\times 10^{-11}\ \text{Wb/m}^2 = 3{\cdot}33\times 10^{-7} \text{ gauss.}$$

(c) $e = EL = 0{\cdot}01 \times 50 = 0{\cdot}5$ volt.

(d) The e.m.f. in the loop will be a maximum when the field at the centre of the loop is zero (see Fig. 137). The field E_1 at one vertical side will be

$$E_1 = E_{\max} \sin 30^\circ = 0{\cdot}005 \text{ volt per metre.}$$

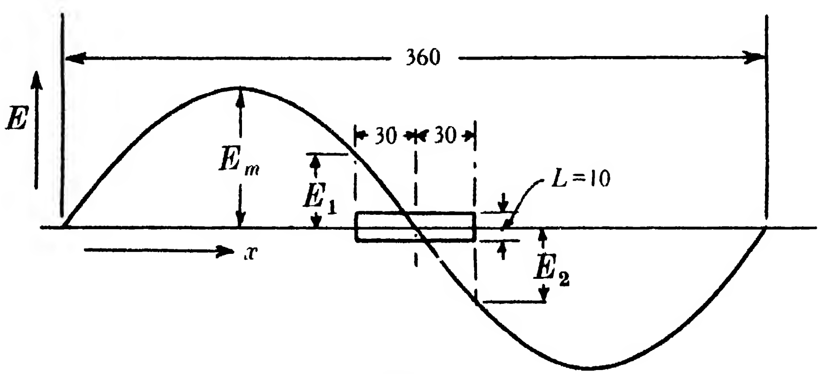


Fig. 137

and the field at the other vertical side

$$E_2 = -0{\cdot}005 \text{ volt per metre,}$$

so that the total e.m.f. in the loop will be

$$e = L(E_1 - E_2) = 0{\cdot}1 \text{ volt.}$$

(e) The maximum value of the energy per cubic metre is

$$\begin{aligned} W_m &= \epsilon_0 E^2 \text{ joules} \\ &= 8{\cdot}854 \times 10^{-12} \times 10^{-4} \\ &= 8{\cdot}854 \times 10^{-16} \text{ joules.} \end{aligned}$$

The mean value, since $\mathscr{E}$ varies sinusoidally, is half the maximum. That is,

$$W = \tfrac{1}{2} W_m = 4{\cdot}427 \times 10^{-16} \text{ joules per cu. metre.}$$

2. Find an expression for the voltage induced in a frame aerial by a plane electro-magnetic wave, and hence describe the directional properties of the frame aerial.

Show how the expression can be simplified if the dimensions of the frame are small compared with the wave-length of the wave.

(London, External B.Sc., 1934.)

3. Devise and explain a wireless receiving system comprising a loop antenna and a vertical-wire antenna which is insensitive to rays of normal polarization (i.e. with electric field vertical), when incident in either of the two directions west-to-east and south-to-north. The width of the loop antenna is to be small compared with the wave-length of the incident rays.

Draw to scale a curve showing how the strength of the signal received depends on the direction of incidence. (Cambridge, B, 1933.)

4. Assuming that the vector potential at the axis of an infinitely long non-magnetic conductor, of circular section (radius a), which carries a current I, is zero, show that, at a point distant r from the axis,

$$A = -\frac{\mu_0 I}{2\pi}\left(\log_e \frac{r}{a} + \frac{1}{2}\right), \quad \text{when } r > a,$$

and

$$A = -\frac{\mu_0 I}{4\pi}\frac{r^2}{a^2}, \quad \text{when } r < a.$$

Note. Use equations 5(21), (22) and (23).

5. A very long ribbon bus-bar has a width $2a$ and its thickness is small. Taking its centre line as the axis of z, and the x-axis in the plane of the ribbon, show that the magnetic field at a point $P(x, y)$, due to a current I flowing in the direction of the z-axis, has components

$$B_x = -\frac{\mu_0 I}{4\pi a}\left[\arctan\frac{a+x}{y} + \arctan\frac{a-x}{y}\right]$$

and

$$B_y = \frac{\mu_0 I}{8\pi a}\left[\log_e \frac{(a+x)^2 + y^2}{a^2} - \log_e \frac{(a-x)^2 + y^2}{a^2}\right].$$

6. Deduce the expression for the force between two parallel conductors of length L, distant D apart (see Ex. 13, Chapter III), by means of the relation $F = I_1 I_2 \frac{dM}{dD}$, where M is given by 5(30).

7. Show that the mutual inductance of two parallel straight wires, of lengths $2L_1$ and $2L_2$, symmetrically situated as shown in Fig. 138, is

$$M = \frac{\mu_0}{2\pi}\left[2L_1 \log_e\left\{\frac{L_1 + L_2 + \sqrt{(L_1+L_2)^2+D^2}}{D}\right\}\right.$$
$$+ (L_1+L_2)\log_e\left\{\frac{L_1+L_2+\sqrt{(L_1+L_2)^2+D^2}}{L_2-L_1+\sqrt{(L_2-L_1)^2+D^2}}\right\}$$
$$\left.+\sqrt{(L_2-L_1)^2+D^2}-\sqrt{(L_1+L_2)^2+D^2}\right].$$

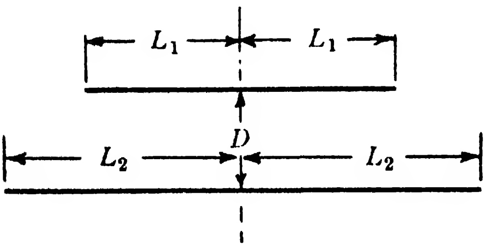


Fig. 138

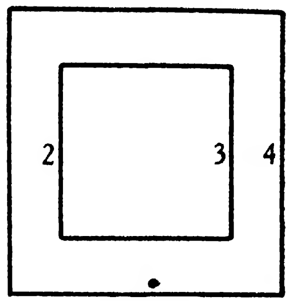


Fig. 139

8. Two square circuits are situated in the same plane, with common centres, and parallel sides (Fig. 139). Show that the mutual inductance is given by

$$M = 4(M_{12} - M_{13}),$$

where M_{12} and M_{13} denote the mutual inductance of the side 1 with sides 2 and 3 respectively.

9. Show that the mutual inductance of two parallel coaxial circles, of radii R_1, R_2 whose planes are distant D apart, is given by

$$M = \mu_0 \sqrt{R_1 R_2}\left\{\left(\frac{2}{k} - k\right)(K) - \frac{2}{k}(E)\right\}$$

where (K) and (E) are the usual elliptic integrals of modulus k, and

$$k^2 = \frac{4R_1R_2}{(R_1+R_2)^2+D^2}.$$

10. A transmitting (vertical) aerial having an effective height of 200 ft. takes a current of 50 amperes (r.m.s.) at a frequency of 480 kilocycles per sec. Calculate:

(1) the radiation resistance of the aerial,
(2) the power radiated,
(3) the aerial efficiency for a total aerial resistance of 50 ohms.

(London, External B.Sc., 1933.)

(*Note*. The conducting surface of the ground can be replaced by an "image" aerial, of the same length as the actual one. Hence the radiated field will consist of the upper half of that of an isolated "dipole" of length 400 ft.) *Ans.* 15·1 ohms, 37·7 kilowatts, 30·2 %.

11. If the effective height of a vertical aerial is equal to $\lambda/2\pi$, where λ is the wave-length, show that the radiation resistance is about 40 ohms.

12. If there is no attenuation of the wave due to absorption of energy by the ground, what is the amplitude of the electric field-strength of the wave due to the aerial of Ex. 10, at a point near the earth's surface 10 miles from the aerial?

Ans. 1·62 millivolts per metre.

13. Discuss the paths of energy flow when a steady current I flows in a return circuit consisting of a resistanceless concentric cable, under a p.d. of V volts. Sketch the lines of energy flow in the dielectric if there is a 5 % resistance drop in each lead.

14. A thin-walled iron cylinder, of length L, mean diameter D, and wall thickness T, is coaxial with a long straight wire carrying a current i. When i changes, show that the component of electric field induced at the centre of the cylinder (on the axis), due to the changing magnetic flux in the cylinder, has intensity given by

$$E=\frac{\mu\mu_0 LT}{\pi D\sqrt{L^2+D^2}}\frac{di}{dt}.$$

[See equation 5(25a), and also 3(15).]

15. Relative to a given observer, a long straight conductor, carrying a steady current, is moving with uniform velocity along its length. An electric charge is moving parallel to the wire with the same velocity, so that the relative velocity between charge and wire is nil.

By means of equations 2(10a) and 5(33) show that the force experienced by the charge, as calculated either by the given observer or a second observer moving with the conductor, is zero.

16. In the case of the simplified wave of Section 4, Part I of this chapter, a homogeneous non-magnetic insulating slab, of dielectric constant K, whose surfaces are parallel to the x-y plane, is moving with uniform velocity w along the x-axis. Show that the velocity of the wave in the insulator, as measured by a stationary observer, is given by

$$v = v_0+\left(1-\frac{1}{K}\right)w,$$

where $v_0=c/\sqrt{K}$, the velocity of the wave when the insulator is at rest, and provided that w^2/c^2 may be neglected.

Solution. Consider a point inside the moving insulator, and stationary relative to the observer who measures w. Let the electric and magnetic fields at this point be E and B, along the z- and y-axes respectively.

Let E_1 be the electic field just outside the surfaces of the insulating slab and E_2 the component due to charges displaced inside the insulator, so that

$$E = E_1 + E_2. \tag{1}$$

Let B_1 be the component of B due to the changing electric field (displacement current), and B_2 the component due to the motion of the displaced charges in the insulator (convection current), so that

$$B = B_1 + B_2. \tag{2}$$

The "inducing" electric intensity acting upon the structure of the moving insulator is

$$E_0 = E_1 + (w \times B_1) = E_1 + wB_1$$

(relative to the moving insulator, B_2 does not exist), so that, from 1(10a),

$$E_2 = -\left(1 - \frac{1}{K}\right)(E_1 + wB_1), \tag{3}$$

and (1) becomes
$$E = \frac{E_1}{K} - \left(1 - \frac{1}{K}\right) wB_1. \tag{4}$$

Then, from 5(8),
$$\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}, \tag{5}$$

while the reciprocal relation, 5(7), becomes

$$\frac{\partial B_1}{\partial x} = \frac{1}{c^2}\frac{\partial E_1}{\partial t}. \tag{6}$$

[The component E_2 causes B_2, and does not contribute to B_1. In the case of a stationary insulator, $E_1 = KE$, and (6) is then identical with 5(7).]

The component B_2 of the magnetic field, is, from 3(6),

$$B_2 = \frac{w \times E_2}{c^2} = \frac{\left(1 - \frac{1}{K}\right) wE_1}{c^2}, \tag{7}$$

neglecting w^2/c^2. Hence, from (2) and (5),

$$\frac{\partial E}{\partial x} = \frac{\partial B_1}{\partial t} + \frac{\left(1 - \frac{1}{K}\right) w}{c^2}\frac{\partial E_1}{\partial t}$$

or
$$\frac{\partial E}{\partial x} = \frac{\partial B_1}{\partial t} + \frac{(K-1)w}{c^2}\frac{\partial E}{\partial t}, \tag{8}$$

from (4) and again neglecting w^2/c^2; while, from (4) and (6),

$$\frac{\partial B_1}{\partial x} = \frac{K}{c^2}\frac{\partial E}{\partial t} + \frac{(K-1)w}{c^2}\frac{\partial B_1}{\partial t}. \tag{9}$$

If the solution of (8) and (9) is of the form

$$E = \alpha \sin p(x - vt),$$

$$B_1 = \beta \sin p(x - vt),$$

then (8) and (9) reduce to

$$-\beta v = \alpha\left\{1+\frac{(K-1)wv}{c^2}\right\}, \tag{10}$$

$$-\frac{K}{c^2}\alpha v = \beta\left\{1+\frac{(K-1)wv}{c^2}\right\}. \tag{11}$$

Multiplying these equations, and taking the square root:

$$\sqrt{K}\frac{v}{c} = 1+\frac{(K-1)wv}{c^2}$$

or

$$\frac{1}{v} = \frac{\sqrt{K}}{c}-\frac{(K-1)}{c^2}w.$$

Now $c/\sqrt{K} = v_0$, the wave velocity when $w = 0$, and $c^2 = Kv_0^2$, so that

$$v = \frac{v_0}{1-\left(\frac{K-1}{K}\right)\frac{w}{v_0}}$$

$$= v_0+\left(1-\frac{1}{K}\right)w, \tag{12}$$

neglecting w^2/v_0^2.

Further, by substituting this result in (10) and (11), we find

$$B_1 = -\frac{E}{v_0}. \tag{13}$$

To find B, the resultant magnetic field, we have, from (4),

$$E_1 = KE+(K-1)wB_1$$

$$= E\left\{K-(K-1)\frac{w}{v_0}\right\}, \tag{14}$$

from (13). And from (7) and (14), neglecting w^2/c^2,

$$B_2 = \frac{(K-1)wE}{c^2},$$

so that

$$B = B_1+B_2 = -E\left\{\frac{1}{v_0}-\frac{(K-1)w}{c^2}\right\}$$

$$= -\frac{E}{c}. \tag{15}$$

(Equation (12) gives the velocity of light in a moving transparent dielectric. The refractive index is given by $\mu = \sqrt{K}$, so we see that the modification in the velocity of light, due to the motion of the transparent body, is equal to $(1-1/\mu^2)$ times the velocity of the body. This fraction is known as *Fresnel's dragging coefficient.*)

17. By means of equations 3(3) and 5(28a), and Poynting's Theorem, 5(18a), show that when a charge q is moving with velocity v ($\ll c$) and acceleration dv/dt, the rate at which energy (other than radiated energy) flows outwards through the surface of a concentric sphere of radius r is given by

$$P = \mu_0\frac{q^2}{6\pi r}v\frac{dv}{dt}.$$

This energy is "stored" in the magnetic field of the moving charge. If the charge is uniformly distributed on a sphere of radius R, show that the rate at which energy flows out of the accelerating sphere is equal to the rate of change of its kinetic energy, where this is given by **3** (52).

EPILOGUE

The experimental discovery of the facts of electro-magnetism in the early part of the nineteenth century was followed by a concerted effort by many mathematical physicists to formulate a comprehensive theory. These workers were divided into two schools, according to their attitude to the philosophical problem of the transmission of effects between material bodies connected by no tangible medium. No disagreement was possible about authenticated experimental facts, but as these facts always consisted of observations on the behaviour of tangible bodies, the question of the transmission of the forces from body to body remained of necessity speculative and hypothetical.

Now the period of which we write is separated from us by the great era of the nineteenth-century mathematical physicists which originated with Maxwell and Kelvin. To them a physical medium was a necessity, and we, being nursed in that great tradition, are apt to forget that some of the best analytical work, following the discoveries, was developed in terms only of current elements and the distance between them. The work of Ampère and F. Neumann, for example, is in perfect accordance with the experimental facts.* Ampère dealt with the forces between current circuits, and Neumann with the induction of e.m.f.'s in a circuit due to changing currents, and in neither of these theories is quantitative use made of the magnetic field. Writing of Ampère's work, Maxwell himself said: "The whole, theory and experiment, seems as if it had leaped, full-grown and full-armed, from the brain of the 'Newton of electricity'. It is perfect in form and unassailable in accuracy; and it is summed up in a formula from which all

* For slowly changing currents.

the phenomena may be deduced, and which must always remain the cardinal formula of electro-dynamics."*

On the other hand, Faraday thought of the action of circuit upon circuit only in terms of magnetic lines of force. Being untrained in mathematics, and finding the idea of "action at a distance" philosophically repugnant, he found this intermediary an absolute necessity in unravelling and formulating the facts, and he seems to have become convinced of its physical reality. We have noted elsewhere, however (p. 292), that he evidently had a feeling that there existed another concept (his electrotonic state, and our vector potential), in terms of which the phenomena could be simply stated, but owing to the necessity of using a concept which had physical significance this idea never took definite form. This was left for Neumann; had Faraday been a mathematician our elementary text-books might to-day be very different.

Clerk Maxwell based his mathematical analysis upon a painstaking study of Faraday's researches, and became deeply impressed by the idea of lines of force. It is interesting to note, however, that his first paper† used the vector potential (he called it the "electrotonic intensity") as a basis for a mathematical expression of the law of induction, but being convinced of the necessity for a physical medium (following Faraday and Kelvin) he later eliminated the vector potential and substituted the magnetic field.

The stroke of genius which lifted Maxwell above his contemporaries was his conception of a changing electric field as an electric current, in that it should be attended by a magnetic field. We have seen how this theory of displacement current leads to a wave-equation, from which is deduced the finite speed of propagation of the effects, but we have also seen that, in order to calculate the magnitude of the fields radiated by a given circuit, we make use of retarded vector and scalar potentials and then forget displacement currents. And we have demonstrated that this procedure is none other than that of forsaking Maxwell for Lorenz, who in 1867 published a theory

* *Treatise on Electricity and Magnetism*, II, Article 528.

† *Trans. Camb. Phil. Soc.* x, p. 27.

of electro-magnetism in which he made no use either of displacement current or magnetic field, but related (by means of his retarded functions) the effects at a distant point directly to (*a*) current elements in which the current is changing, and (*b*) the instantaneous distribution of charges. By suitable analysis, Maxwell's theory reduces to that of Lorenz.

The fundamental difference in the two viewpoints was not one of fact, but of hypothesis. The viewpoint of Maxwell necessitated the existence of a material medium, or aether, which could transmit energy and disturbances in much the same way as an elastic solid, while the denial of such a medium meant the transmission of energy in a form comparable with that when a cricket ball breaks a window. Let us quote Maxwell:*

> Now we are unable to conceive of propagation in time, except either as the flight of a material substance through space, or as the propagation of a condition of motion or stress in a medium already existing in space. In the theory of Neumann,† the mathematical conception called Potential, which we are unable to conceive as a material substance, is supposed to be projected from one particle to another, in a manner which is quite independent of a medium, and which, as Neumann has himself pointed out, is extremely different from that of the propagation of light....
>
> But in all of these theories the question naturally occurs: if something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? If this something is the potential energy of the two particles, as in Neumann's theory, how are we to conceive this energy as existing in a point of space, coinciding neither with the one particle nor with the other?

What an extraordinarily interesting argument in the light of modern physics! Although it is not known whether there is a continuous distribution of energy in a wave, it is certain that radiated energy is emitted and absorbed in indivisible "quanta", of content hf, where h is Planck's Constant, and f is the frequency of the radiation. Electro-magnetic radiation *does* indeed behave, apparently, like cricket balls flying through

* *Treatise*, II, Article 866.

† C. Neumann, *Mathematische Annalen*, I, p. 317.

the air: it has a dual personality, sometimes particle, sometimes wave.

All this seems a far cry from electrical engineering, and yet it is extremely relevant to the problem of teaching the fundamentals. The day seems to be long past when physicists held an understanding of the "mechanism" of electromagnetic phenomena to be possible, in terms of the physical properties of a universal medium; all we can do, to-day, is to accept the physical facts and confess our ignorance of the underlying mechanism. What we lose in self-conceit we gain in clarity of thought. We must, however, acknowledge our great debt to the master-minds of the last century; for the field concepts of Faraday and Maxwell, if regarded as mathematical vectors and not as physical realities, are of extraordinary value in simplifying our theory and our calculations.

These field concepts, moreover, have proved so fruitful in the past, and have played such an important rôle in the development of modern physics, that it would be surprising to see them pass irrevocably from the scene. There are many who, like Einstein, put their faith in "field physics" as the road which may lead ultimately to a satisfactory synthesis of the physical world. The gap of ignorance that we now leave between charge and charge should not dismay us, for it is but a small part of the unfathomed mystery of experience which embraces all things, measurable or not, of which we are conscious. If we lose a comfortable but blinding faith we gain, one may hope, a humble open-mindedness which should prepare us to welcome, as they are uncovered, new fragments of the truth.

APPENDIX I

THE M.K.S. SYSTEM OF UNITS

The c.g.s. system of electrical and magnetic units had its first official approval in a report of a British Association Committee in 1873. In this report the names *dyne* (unit force), *erg* (unit work or energy) and *erg per second* (unit power) were advocated. In 1881 the First Electrical Congress, in Paris, ratified the c.g.s. electro-

magnetic system as fundamental, and adopted five practical units (the ohm, volt, ampere, coulomb and farad) to be derived therefrom. Practical units adopted since then are: the *joule* and *watt* (1889), the *henry* (1893), the *weber* ( ), and the *newton** ( ).

Maxwell (Section 629, Vol. II, of his *Treatise*) pointed out that the practical units could be included in an *absolute* system having 10^{-11} gramme as the unit of mass, 10^7 metres (or an "earth quadrant") as the unit of length, and the second as the unit of time, still maintaining unit value for μ_0. These units, of course, are quite unpractical, but in 1901 G. Giorgi demonstrated that if the permeability of space (μ_0) were taken as 10^{-7}, instead of unity, then the practical units formed an absolute system, taking the *metre* as unit length, the *kilogramme* as unit mass, and the *second* as unit time.†

In 1904 Ascoli, a colleague of Giorgi's, pointed out‡ that there existed an indefinite number of such "practical" systems, governed by the rule $(2L+m) = 7$, where 10^L cm. is the unit of length, and 10^m gm. is the unit of mass. Of these, however, only Maxwell's suggested system ($L = 9$, $m = -11$) maintained μ_0 as unity, while only Giorgi's m.k.s. system ($L=2$, $m=3$) included the international standards of length and mass (the metre and the kilogramme) maintained by the International Bureau of Weights and Measures, near Paris.

Giorgi's system gradually gained favour, and the culminating point was reached in 1935 when, at Scheveningen-Brussels, the International Electrotechnical Commission unanimously adopted it as an absolute system, incorporating all the practical units.§ In the same year the International Committee of Weights and Measures, meeting at Sèvres, decided that the absolute system of electrical units should take the place of the international system, as the basis of standard measurements, and that this substitution should take place on 1 January .

In connection with this action of the I.E.C., two important issues were left over for decision at some future date.

THE FOURTH FUNDAMENTAL UNIT

Having adopted the metre, the kilogramme, and the second as fundamental units, the further adoption of any one of the practical electrical units completes the system.

From the point of view of a logical *theory* (i.e. from the point of view of this book), the *coulomb* (unit of charge) has distinct claims to be regarded as the fundamental electrical unit, and is so regarded in the dimensional equations in Appendix II. From practical considerations, however, the coulomb is unsuitable as a basis for international standards of measurement, since we cannot keep a "standard coulomb" in our standardizing laboratories.

At the meeting, Dr Kennelly and Prof. Lombardi were asked to obtain the views on this matter of the Electrical Advisory Committee of the International Conference on Weights and Measures, and of the Symbols, Units and Nomenclature (S.U.N.) Committee of the International Union of Pure and Applied Physics. The I.E.C. suggested that one of the seven units—coulomb, ampere, volt, ohm, henry, farad and weber—should be adopted, but the bodies consulted evidently felt that an arbitrary choice of one of these might be construed as a dogmatic assertion which might very well prove detrimental to the future development of physical knowledge. They consequently recommended that the m.k.s system should be completed merely by taking the permeability of free space as equal to 10^{-7}. At the meeting of the I.E.C. held at Torquay in June the Advisory Committee on Electric and Magnetic Magnitudes and Units (E.M.M.U.) accepted this recommendation in the following resolution, which was afterwards ratified by the Committee of Action:

"The Committee, noting the concordant replies of Comité Consultatif d'Electricité and of the S.U.N. Committee of the International Union of Pure and Applied Physics as to the choice of a fourth unit in the Giorgi (M.K.S.) system agrees to recommend as the connecting link between the electrical and mechanical units, the permeability of a free space with the value of $\mu_0 = 10^{-7}$ in the unrationalized system, or $4\pi 10^{-7}$ in the rationalized system.

"The Committee recognizes that any one of the following practical units, ohm, ampere, volt, henry, farad, coulomb, weber, already in use may equally serve as the fourth fundamental unit, because it is possible to derive each unit and its dimensions from any four others mutually independent."*

In 1950 a decision was reached. In that year the I.E.C. met in Paris and adopted the ampere as the fourth preferred fundamental

unit in the Giorgi system. The International Union of Pure and Applied Physics adopted the ampere as the fourth fundamental unit in 1951.

The Rationalized System

In the classical c.g.s. system of units, field relations having spherical symmetry do not contain the factor 4π, whereas this factor, or a multiple of π, occurs in field relations in cases of plane geometry such as the capacitance of a parallel-plate capacitor. Further, the circuital law of the magnetic field appears in the form: magnetomotive force $= 4\pi IN$. These results arise directly from the foundation of the c.g.s. systems in the simplest possible form of the inverse square law for the force between two point charges or poles, e.g. $F = q_1 q_2/r^2$.

In the m.k.s. system of units Coulomb's law cannot be expressed in such a simple form since a proportionality factor, which we have called the *primary electric constant*, is necessary. If we introduce this factor in such a way that Coulomb's law becomes

$$F = \frac{q_1 q_2}{\epsilon_0' r^2}, \qquad (a)$$

where ϵ_0' is the primary electric constant, we then derive a system of electromagnetic equations in which the factor π occurs in the same context as in the c.g.s. system. If, however, we express the law in the form

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}, \qquad (b)$$

the equations contain the factor π in cases of spherical or cylindrical symmetry and not otherwise. The system based on (a) is called the "unrationalized" system while that based on (b) is the "rationalized" system which we have used in the text.

At the meeting of the I.E.C. in 1935 there was so much difference of opinion as to the respective merits of the two systems that a decision was deferred. In the first edition of this book, published in 1939, the unrationalized form was used since this provides the simplest introduction to the use of the m.k.s. system for those who are used to the classical c.g.s. systems, and full instructions were included for converting all the relevant equations into the rationalized form. The position has now been clarified, however in favour of the rationalized system.

The first definite step towards the decision to adopt the rationalized system was taken in July 1950, when Technical Committee

No. 24 of the I.E.C. (the Advisory Committee on Electric and Magnetic Magnitudes and Units), meeting in Paris, recommended that "total rationalization" should be adopted for the Giorgi system. This decision was reached only after "long and laborious discussions"*, and a Committee of Experts was appointed to undertake a more extensive study of the rationalization process. The recommendation received general support in the profession†, but final approval by the I.E.C. did not come until the meeting in Munich,‡ It was then agreed that rationalization of the electromagnetic equations should be characterized by certain "principal equations" (including the electrostatic form of the Theorem of Gauss and the magnetic circuital law) "whose form remains the same whether the symbols are regarded as representing the physical quantities coming into play or are regarded as representing their numerical values". The latter statement removed a source of confusion and controversy the details of which need not concern us here.

Quantities affected by rationalization

Only eight basic quantities in electromagnetism are affected by rationalization, as shown in the following table.

Quantity	In the rationalized system	The equivalent in the unrationalized system
Primary electric constant	ϵ_0	$\epsilon_0' = 4\pi\epsilon$
Primary magnetic constant	μ_0	$\mu_0' = \mu_0/4\pi$
Electric flux	ψ	$\psi' = 4\pi\psi$
Electric flux-density	D	$D' = 4\pi D$
Magnetomotive force	m	$m' = 4\pi m$
Magnetic field intensity	H	$H' = 4\pi H$
Magnetic pole	$\mathcal{M}$	$\mathcal{M}' = \mathcal{M}/4\pi$
Intensity of magnetization	M	$M' = M/4\pi$

With the aid of this table the reader can readily convert any electromagnetic equation in the text from the rationalized to the unrationalized form. All other electric and magnetic quantities are unaffected by rationalization.

New names of units

The existing practical units basic to the theory of the electric circuit were established long before the introduction of the rationalized m.k.s. system, and had been given the names of eminent scientists. Thus the coulomb, ampere, ohm, volt, henry and farad fitted naturally into the new system, but certain important quantities, particularly mechanical force and the magnetic field measures, had previously been used in c.g.s. units and had no distinctive names in the m.k.s. system. The following table gives the dates of I.E.C. recommendations and decisions to meet this need.*

1935	unit of magnetic flux: the *weber* unit of frequency: the *hertz* (this is of course not confined to the m.k.s. system) unit of conductance: the *siemens*
1938	unit of mechanical force: proposal of the *newton*
1950	unit of mechanical force: the *newton* adopted
1954	unit of flux-density: proposal of the *tesla*
1956	unit of flux-density: the *tesla* adopted

The I.E.C. still has under consideration the question of finding names for further units, such as that of magnetic field intensity which at present is designated as the "ampere per metre". It is a little difficult to understand why personal names should be required for the magnetic field measures but not for those of the electric field. The "volt per metre" and the "coulomb per sq. metre" seem to be in no danger, at present, of being replaced, but perhaps this is because the c.g.s. units for these quantities have no distinctive personal names. It may also be noted that the I.E.C. is at last reconsidering the terms "permeability of free space" and "permittivity of free space" and their designations μ_0 and ϵ_0.

Definitions of Electrical Units

The following definitions were adopted by the Eleventh General Conference of Weights and Measures, in October 1960. These definitions are not meant to affect in any way the freedom, in scientific work, to define a particular unit in any way consistent with the fundamental physical laws, and were laid down "as no more than simple verbal statements sufficient to fix the numerical values of the constants in the equations."

* I.E.C. Publication 164 (Geneva, 1964), p. 37.

The ampere (unit of electric current) is the constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular sections, and placed 1 metre apart in a vacuum, will produce between these conductors a force equal to 2×10^{-7} newton per metre of length.

The volt (unit of difference of potential and of electromotive force) is the difference of electric potential between two points of a conducting wire carrying a constant current of 1 ampere, when the power dissipated between these points is equal to 1 watt.

The ohm (unit of electric resistance) is the electric resistance between two points of a conductor when a constant difference of potential of 1 volt, applied between these points, produces in the conductor a current of 1 ampere, the conductor not being the seat of any electromotive force.

The coulomb (unit of quantity of electricity) is the quantity of electricity transported in 1 second by a current of 1 ampere.

The farad (unit of electric capacitance) is the capacitance of a capacitor between the plates of which there appears a difference of potential of 1 volt when it is charged by a quantity of electricity of 1 coulomb.

The henry (unit of electric inductance) is the inductance of a closed circuit in which an electromotive force of 1 volt is produced when the electric current in the circuit varies uniformly at the rate of 1 ampere per second.

The weber (unit of magnetic flux) is the magnetic flux which, linking a circuit of 1 turn produces in it an electromotive force of 1 volt as it is reduced to zero at a uniform rate in one second.

Summaries of the M.K.S. and C.G.S. Systems

The reader should now turn to the table on p. 315, which gives the c.g.s. (electromagnetic) equivalents of rationalized m.k.s. units.

APPENDIX II

DIMENSIONS OF ELECTRICAL AND MAGNETIC QUANTITIES

The familiar concepts of mechanics are all functions of three physical "dimensions" (i.e. primary concepts): length (L), mass (M), and time (T). The dimensions of any derived concept can then be given by "dimensional formulae", as follows:

Concept	Dimensional formula
Velocity, $v = \dfrac{\text{length}}{\text{time}}$	LT^{-1}
Acceleration, $\alpha = \dfrac{\text{velocity}}{\text{time}}$	LT^{-2}
Force, F = (mass) × (acceleration)	LMT^{-2}
Work and energy, W = (force) × (length)	L^2MT^{-2}
Power, $P = \dfrac{\text{energy}}{\text{time}}$	L^2MT^{-3}
etc.	

Now the dimensions of electrical and magnetic quantities cannot be given in terms of L, M and T only, and an additional primary concept, or dimension, is needed. In text-books of physics it is usual to find either ϵ_0 or μ_0 (the primary electric and magnetic constants) used for the additional primary concept, which results in the expressions for the dimensions of electro-magnetic quantities containing fractional powers of the primary dimensions. Further, two distinct sets of expressions are obtained, one based on electro-static, and one based on electro-magnetic, units.

It appears to the author to be more reasonable to take electric charge or quantity (Q), as the additional primary concept,* for by so doing the expressions for dimensions become simpler and consequently easier to use. Further, electric charge appears to be a very fundamental concept, possibly more fundamental than mass, whereas ϵ_0 and μ_0 are merely connecting links between "secondary" concepts (i.e. $D=\epsilon_0 E$ and $B=\mu_0 H$) whose value rests in their use as an aid to our analysis and calculation of electrical phenomena. The continued use of μ_0 and ϵ_0 as primary concepts is really no more

than a memorial to a theory long since dead: that of the mechanical aether of the nineteenth-century physicists.

The main use of these dimensional formulae is in checking the final formulae of any theoretical development. For example, if the formula is that for a force, then the dimensions of all the factors in the formula must combine to form those of a force, and if this is not so a mistake has been made. But the checking of dimensions in this way cannot disclose any error in numerical coefficients, for these are "dimensionless", or "numerics", and do not enter into the dimensional formulae.

Adopting "Q" as the fourth primary concept, the dimensional formulae of the chief electrical and magnetic quantities are developed as follows.

The Dimensions of Electrical and Magnetic Quantities in terms of the primary dimensions L, M, T and Q

	Concept	Dimensions
(1)	Charge, quantity, and displacement, q and ψ	Q
(2)	Displacement density, $D = \dfrac{Q}{\text{area}}$	$L^{-2}Q$
(3)	Current, $I = \dfrac{Q}{\text{time}}$	$T^{-1}Q$
(4)	Electric field intensity, $E = \dfrac{\text{force}}{\text{charge}}$	$LMT^{-2}Q^{-1}$
(5)	E.m.f. or p.d. (e or V) $= E \times$ (length)	$L^2MT^{-2}Q^{-1}$
(6)	Resistance, $R = \dfrac{V}{I}$	$L^2MT^{-1}Q^{-2}$
(7)	Capacitance, $C = \dfrac{Q}{V}$	$L^{-2}M^{-1}T^2Q^2$ $(=\epsilon_0 L)$
(8)	Magnetic flux, $\phi =$ (e.m.f.) $\times$ (time)	$L^2MT^{-1}Q^{-1}$
(9)	Magnetic flux-density, $B = \dfrac{\text{flux}}{\text{area}}$	$MT^{-1}Q^{-1}$
(10)	M.m.f., $m = I \times$ (numeric)	$T^{-1}Q$
(11)	M.m.f. gradient, $H = \dfrac{\text{m.m.f.}}{L}$	$L^{-1}T^{-1}Q$
(12)	Inductance $= \dfrac{\text{flux}}{\text{current}}$	L^2MQ^{-2} $(=\mu_0 L)$
(13)	Primary electric constant, $\epsilon_0 = \dfrac{D}{H}$	$L^{-3}M^{-1}T^2Q^2$
(14)	Primary magnetic constant, $\mu_0 = \dfrac{B}{H}$	LMQ^{-2}

Examples of the Combination of Dimensions

(A) *Energy.*

$W = \frac{1}{2}LI^2$ [eq. 3(38)]

Dimensions are: $L^2MQ^{-2}T^{-2}Q^2 \qquad = L^2MT^{-2}$

$W = \frac{1}{2}(ED)$ (volume) [eq. 1(21)]

Dimensions are: $LMT^{-2}Q^{-1}L^{-2}QL^3 \qquad = L^2MT^{-2}$

$W = \frac{1}{2}(HB)$ (volume) [eq. 3(49)]

Dimensions are: $L^{-1}T^{-1}QMT^{-1}Q^{-1}L^3 \qquad = L^2MT^{-2}$

(B) *Force.*

$F = IN\dfrac{d\phi}{dx}$ [eq. 2(4), N is a numeric]

Dimensions are: $\dfrac{T^{-1}QL^2MT^{-1}Q^{-1}}{L} \qquad = LMT^{-2}$

$F = BLI$ [eq. 2(9)]

Dimensions are: $MT^{-1}Q^{-1}LT^{-1}Q \qquad = LMT^{-2}$

(C) *Power.*

$P = I^2R.$

Dimensions are: $T^{-2}Q^2L^2MT^{-1}Q^{-2} \qquad = L^2MT^{-3}$

etc., etc.

(D) *The product* $\epsilon_0\mu_0$.

We have $\mu_0 = 4\pi \times 10^{-7}$,

and $\epsilon_0 = \dfrac{10^7}{4\pi c^2}$,

where c is the velocity of light,

so that $\epsilon_0\mu_0 = \dfrac{1}{c^2}$.

The dimensions of the product are

$$L^{-3}M^{-1}T^2Q^2LMQ^{-2} = L^{-2}T^2,$$

i.e. the dimensions of $\dfrac{1}{(\text{velocity})^2}$.

Comparative Table of Dimensions in Three Systems

The dimensional equations of the chief electrical and magnetic quantities are given in the following table in the ($LMTQ$) system, together with those in the two conventional systems. These latter are given for reference and comparison, and are not recommended for practical use.

Quantity	Dimensional formulae		
	Electro-static system (LMT and ϵ)	Electro-magnetic system (LMT and μ)	$LMTQ$ system
Charge, q (and displacement)	$L^{\frac{3}{2}}MT^{\frac{1}{2}-1}\epsilon^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$	Q
Displacement density	$L^{-\frac{1}{2}}M^{\frac{1}{2}}Y^{-1}\epsilon^{\frac{1}{2}}$	$L^{-\frac{3}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$	$L^{-2}Q$
Current	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$T^{-1}Q$
Electric field intensity	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$	$LMT^{-2}Q^{-1}$
E.m.f. and p.d.	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\epsilon^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$	$L^{2}MT^{-2}Q^{-1}$
Resistance	$L^{-1}T\epsilon^{-1}$	$LT^{-1}\mu$	$L^{2}MT^{-1}Q^{-2}$
Capacitance	$L\epsilon$	$L^{-1}T^{2}\mu^{-1}$	$L^{-2}M^{-1}T^{2}Q^{2}(L\epsilon_0)$
Magnetic flux	$L^{\frac{1}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$L^{2}MT^{-1}Q^{-1}$
Flux density	$L^{-\frac{3}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$MT^{-1}Q^{-1}$
M.m.f. gradient	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\epsilon^{\frac{1}{2}}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{-1}T^{-1}Q$
Inductance	$L^{-1}T^{2}\epsilon^{-1}$	$L\mu$	$L^{2}MQ^{-2}(L\mu_0)$
Vector potential	$L^{-\frac{1}{2}}M^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$LMT^{-1}Q^{-1}$
ϵ_0	ϵ	$L^{-2}T^{2}\mu^{-1}$	$L^{-3}M^{-1}T^{2}Q^{2}$
μ_0	$L^{-2}T^{2}\epsilon^{-1}$	μ	LMQ^{-2}
K and μ	—	—	Numeric

THE M.K.S. AND C.G.S. SYSTEMS

M.K.S. (Rationalized)	C.G.S. (Electro-magnetic)
$\mu_0 = 4\pi \times 10^{-7}$ henry/metre	$\mu_0 = 1$
$\epsilon_0 = \dfrac{1}{c^2\mu_0} = 8{\cdot}854 \times 10^{-12}$ farad/metre	$\epsilon_0 = \dfrac{1}{c^2}$
$c = 2{\cdot}998 \times 10^8$ m/s, $c^2 = 8{\cdot}987 \times 10^{16}$	$c = 2{\cdot}998 \times 10^{10}$ cm/s, $c^2 = 8{\cdot}987 \times 10^{20}$

	M.K.S. Units	Equivalent in C.G.S. Units
Length, L	1 metre	10^2 cm. (3·2809 ft.)
Mass, M	1 kilogramme	10^3 gm. (2·2046 lb.)
Time, T	1 second	1 second
Velocity, v	1 metre per sec.	10^2 cm. per sec.
Angular velocity, ω	1 radian per sec.	1 radian per sec.
Acceleration, α	1 metre per sec.^{2}	10^2 cm. per sec.^{2}
"g"	9·81 metres per sec.^{2}	981 cm. per sec.^{2}
Force, F	1 newton	10^5 dynes (0·2247 lb. wt.)
Pressure, p	1 newton per sq. metre	10 dynes per sq. cm (0·02088 lb. per sq. ft.)
Work and energy, W	1 joule	10^7 ergs (0·7373 ft.-lb.)
Power, P	1 watt	10^7 ergs per sec
Torque, τ (= P/ω)	1 newton-metre (1/9·81 kilogramme-metre)	10^7 dyne-cm. (0·7373 lb.-ft.)
Moment of inertia, J	1 kilogramme-(metre)^2	10^7 gm.-(cm.)^2 (23·73 lb.-ft.^{2})
Electric charge, q	1 coulomb	10^{-1} ab-coulomb
Current, I	1 ampere	10^{-1} ab-ampere
Current density, J	1 ampere per sq. metre	10^{-5} ab-ampere per sq. cm.
Resistance, R	1 ohm	10^9 ab-ohms
Resistivity, ρ	1 ohm-metre	10^{11} ab ohm-cm. (10^2 ohm-cm.)
Conductance, G	1 mho* (siemens*)	10^{-9} ab-mho
Conductivity, γ	1 mho per metre	10^{-11} ab mho per cm. (10^{-2} mho per cm.)
Electric flux, ψ	1 coulomb	$4\pi/10$ c.g.s. e-m units
Electric flux-density, D	1 coulomb per sq. metre	$4\pi/10^5$ c.g.s. e-m units
Electric field intensity, E	1 volt per metre	10^6 ab-volts per cm.
E.m.f. and p.d., E, e and V	1 volt	10^8 ab-volts
Capacitance, C	1 farad	10^{-9} ab-farad
Magnetic flux, ϕ	1 weber (volt-second)	10^8 maxwells
Magnetic flux-density, B	1 weber per sq. metre (tesla)	10^4 gauss (maxwells per sq. cm.)
Inductance, L and M	1 henry	10^9 ab-henries
M.m.f., m	1 ampere-turn	$4\pi \times 10^{-1}$ gilbert
M.m.f. grad, H (magnetic field intensity)	1 ampere-turn per metre	$4\pi \times 10^{-3}$ oersted (gilberts per cm.)
Vector potential, A	1 weber per metre (volt-second per metre)	10^6 maxwells per cm.
Dielectric constant, K (relative permittivity)	numeric	= numeric
Relative permeability, μ	numeric	= numeric

* Both these names are in current use in different countries, but siemens has been internationally adopted.

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